Fuzzy Mathematics, Graphs, and Similarity Measures

Fuzzy Mathematics, Graphs, and Similarity Measures

Analysis and Application Across Global Challenges

John N. Mordeson

Sunil Mathew

ELSEVIER

ACADEMIC PRESS

An imprint of Elsevier

Academic Press is an imprint of Elsevier
125 London Wall, London EC2Y 5AS, United Kingdom
525 B Street, Suite 1650, San Diego, CA 92101, United States
50 Hampshire Street, 5th Floor, Cambridge, MA 02139, United States

ISBN: 978-0-443-33949-3

For information on all Academic Press publications
visit our website at https://www.elsevier.com/books-and-journals

Publisher: Peter Linsley
Editorial Project Manager: Joshua Mearns
Publishing Services Manager: Deepthi Unni
Production Project Manager: Prasanna Kalyanaraman
Cover Designer: Christian Bilbow

Typeset by VTeX

J. N. Mordeson would like to dedicate the book to victims of violence.

Sunil Mathew would like to dedicate the book to his family.

Contents

About the authors

Dr. John N. Mordeson is Professor Emeritus of Mathematics at Creighton University. He received his B.S., M.S., and Ph.D. from Iowa State University. He is a member of Phi Kappa Phi. He has published 20 books and 250 journal articles. He is on the editorial board of numerous journals. He has served as an external examiner of Ph.D. candidates from India, South Africa, Bulgaria, and Pakistan. He has refereed for numerous journals and granting agencies. He is particularly interested in applying mathematics of uncertainty to combat the problem of human trafficking.

Dr. Sunil Mathew is a Faculty Member in the Department of Mathematics, NIT Calicut, India. He has acquired his masters from St. Joseph's College Devagiri, Calicut, and Ph.D. from National Institute of Technology Calicut in the area of Fuzzy Graph Theory. He has published more than 150 research papers and written 9 books. He is a member of several academic bodies and associations. He is editor and reviewer of several international journals. He has an experience of 25 years in teaching and research. His current research topics include fuzzy graph theory, bio-computational modeling, graph theory, fractal geometry, and chaos.

Preface

The world is faced with many serious challenges. In this book, we use mathematics of uncertainty to study these challenges. We consider the challenges human trafficking, cybersecurity, terrorism, mistreatment of women and children, space debris, and scamming. We provide Exercises and an instructor's manual in case the book is used for a college course on global issues.

In Chapter 1, we provide the notation and mathematical concepts needed for the book. The first part of the book rests heavily on fuzzy similarity measures. We provide the basis ideas of fuzzy similarity measures and fuzzy implication operators. We also lay the foundation to fuzzy graph theory, which is used to examine human trafficking and other global problems.

In Chapter 2, we discuss how theoretical results from one family of fuzzy sets can be immediately carried over to another by the use of lattice isomorphisms. We show, however, that these families of fuzzy sets can arise natural in applications and that results of the application may not be carried over from one family to another.

In Chapter 3, we consider cyber security. We determine fuzzy similarity measures of several issues with respect to country rankings. These issues include the risk to cyber crime. Statista collected data used in NordVPN's analysis and approved the methodology used to create the Cyber Risk Index.

In Chapter 4, we consider terrorism and bioterrorism. We examine the impact of terrorism and its effect on countries. We also find fuzzy similarity measures concerning issues related to terrorism and bioterrorism. We examine the relationship of Interpol global policy goals and sustainable development goals.

In Chapter 5, we discuss how fuzzy implication operators can be used to determine fuzzy similarity measures involving the issues of health security and political risk. We also consider a country's vulnerability and exposure to natural disaster.

In Chapter 6, we once again use fuzzy similarity to examine country health issues. The Global Health Security Index has stated that all countries remain dangerously unprepared for future epidemic and pandemic threats. We use fuzzy implication operators as a basis of our study. For ease of reading, a small amount of results from Chapter 5 is repeated.

In Chapter 7, we consider the mistreatment of women and children. There are many types of technology-facilitated violence. Studies confirm a high prevalence rate against women and girls. We study this situation using fuzzy implication operators.

In Chapter 8, we consider space debris and sustainability. At the current rate of expansion, an environmental crisis in space will occur unless we act now. We consider issues involving sustainability and space. In particular, the connection between space debris and artificial intelligence is examined.

In Chapter 9, we consider telecommunications. We provide the ranking of countries concerning issues of internet speed. We also consider the problem of spam and scam.

Chapter 10 discusses two topological indices related with fuzzy graphs, namely eccentric connectivity index and modified eccentric connectivity index. Both indices are evaluated for most of the important fuzzy graph structures. An application of the index involving human trafficking chains is also presented.

Chapter 11 deals with another fuzzy graph index known as neighborhood connectivity index, which can be used in quality of service problems and routing problems. This index is provided for a number of fuzzy graph structures. Neighborhood connectivity index of different types of products of fuzzy graphs is also discussed.

In Chapter 12, we present sigma index and average sigma index of a fuzzy graph. A number of results evaluating these indices for different categories of fuzzy graphs and comparison of these indices with first and second Zagreb index are also provided. An algorithm for the computation of sigma index is given. A related application in the field of financing is discussed at the end.

Chapter 13 gives another couple of indices namely Banhatti indices of first and second kind. Index values are obtained for paths, cycles, trees, complete fuzzy graphs, etc. Banhatti indices of graphs obtained by fuzzy graph operations are also discussed. Related algorithms and applications are also discussed.

Chapter 14 presents a new type of connectivity parameter in fuzzy graphs called generalized cycle connectivity. In classical cycle connectivity, we use only strong cycles, whereas here every cycle is accounted. This will help us in the modeling of cyclic flow through various types of networks in a very different way. Most of the major fuzzy graph structures have been examined and their generalized cycle connectivity have been evaluated. Generalized cyclic cut vertices, bridges, generalized cyclic edge, and vertex connectivity are also discussed in detail. Cyclic stability and applications are also provided toward the end.

Chapter 15 discusses another important fuzzy graph connectivity parameter called average fuzzy edge connectivity. We have presented the evaluation of average edge connectivity for most of the fuzzy graph classes. An application related to stress in youth is also provided.

John N. Mordeson
Omaha, NE, United States

Sunil Mathew
Calicut, India

Acknowledgments

The authors are grateful to the editorial board and production staffs of Elsevier, especially to Peter Linsley. The authors are indebted to the journals New Mathematics and Natural Computation, Information Sciences, Transaction on Fuzzy Sets and Systems, Fuzzy Sets and Systems, Fuzzy Information and Engineering, Iranian Journal of Fuzzy Systems, Journal of Malani Mathematical Research, Journal of Algebraic Hyperstructures and Logical Algebras, IEEE Transactions on Fuzzy Systems.

John N. Mordeson
Omaha, NE, United States

Sunil Mathew
Calicut, India

CHAPTER

Preliminaries[*] 1

In this chapter, we provide the notation and concepts needed for our book. We assume the reader is familiar with basic set theory. We first consider notation. We let $\wedge$ denote minimum or infimum and $\vee$ denote maximum or supremum. $\mathbb{N}$ denotes the positive integers. We let $[0, 1]$ denote the closed unit interval. Let A be a subset of a set X. Then $X\backslash A$ (or A^c if the context is clear) denotes the complement of A in X. We let $|A|$ denote the cardinality of A and A^n denote the Cartesian cross product of A n times, where $n \in \mathbb{N}$. We let $\mathcal{P}(X)$ denote the power set of X.

1.1 Fuzzy sets

In 1965, Lotfi A. Zadeh [129] introduced the concept of a fuzzy set and a fuzzy logic.

Let X be a set and A a subset of X. The **characteristic function** of A is the function χ of X into $\{0, 1\}$ defined by $\chi(x) = 1$ if $x \in A$ and $\chi(x) = 0$ if $x \notin A$. The characteristic function can be used to indicate either members or nonmembers of A. This notion can be generalized in a way that introduces the notion of a fuzzy subset of X.

Definition 1.1.1. A fuzzy subset μ of X is a function of X into the closed interval $[0, 1]$.

Let μ be a fuzzy subset of a set X. For all $x \in X$, $\mu(x)$ can be thought of as the degree of membership of x in μ. We sometimes use the notation μ_A for a fuzzy subset of X, where A is thought of as a fuzzy set and μ_A gives the grade of membership of elements of X in A. At times, A may be merely a description of a fuzzy subset μ of X.

We let $\mathcal{FP}(X)$ denote the **fuzzy power set** of X, i.e., the set of all fuzzy subsets of X.

Definition 1.1.2. Let μ be a fuzzy subset of X.

(1) Let $\alpha \in [0, 1]$. Define $\mu_a = \{x \in X | \mu(x) \geq \alpha\}$. We call μ_α and a-**cut** or a α-**level set**.

(2) The **support** of μ is defined to be the set $\text{Supp}(\mu) = \{x \in X | \mu(x) > 0\}$.

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00009-X

Definition 1.1.3. Let μ and ν be fuzzy subsets of X. Define μ^c, $\mu \cap \nu$, and $\mu \cup \nu$ as follows: $\forall x \in X$,

$$\mu^c(x) = 1 - \mu(x),$$
$$(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x),$$
$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x).$$

Then μ^c is called the **(standard) complement** of μ, $\mu \cap \nu$ the **intersection** of μ and ν, and $\mu \cup \nu$ the **union** of μ and ν.

We can extend the notions of intersection and union to a family of fuzzy subsets of X. Let $\{\mu\}_{i \in I}$ be a family of fuzzy subsets of X, where I is an index set. Define $\cap_{i \in I}$ and $\cup_{i \in I}$ as follows: $\forall x \in X$,

$$(\cap_{i \in I} \mu_i)(x) = \wedge_{i \in I} \mu_i(x),$$
$$(\cup_{i \in I} \mu_i)(x) = \vee_{i \in I} \mu_i(x)$$

Thus if I is a finite set, say $I = \{1, 2, ..., n\}$, then $(\cap_{i \in I} \mu_i) = \mu_1 \cap \mu_2 \cap ... \cap \mu_n$ and $(\cup_{i \in I} \mu_i) = \mu_1 \cup \mu_2 \cup ... \cup \mu_n$. In this case, we sometimes write $(\cap_{i \in I} \mu_i)(x) = \mu_1(x) \wedge \mu_2(x) \wedge ... \wedge \mu_n(x)$ and $(\cup_{i \in I} \mu_i)(x) = \mu_1(x) \vee \mu_2(x) \vee ... \vee \mu_n(x)$.

The intersection of two fuzzy subsets of a set is specified in general by a binary operation on the unit interval; that is, a function of the form

$$i : [0, 1] \times [0, 1] \to [0, 1].$$

Definition 1.1.4. A fuzzy intersection (t**-norm**) is a binary relation i on the unit interval that satisfies the following properties: $\forall a, b, d \in [0, 1]$:

(1) $i(a, 1) = a$ (boundary condition);

(2) $b \leq d$ implies $i(a, b) \leq i(a, d)$ (monotonicity);

(3) $i(a, b) = i(b, a)$ (commutativity);

(4) $i(a, i(b, d)) = i(i(a, b), d)$ (associativity).

Example 1.1.5. Let $a, b \in [0, 1]$.

Standard intersection: $i(a, b) = a \wedge b$,

Algebraic product: $i(a, b) = ab$,

Bounded difference: $i(a, b) = 0 \vee (a + b - 1)$.

The general fuzzy union of two fuzzy subsets is specified by a function $u : [0, 1] \times [0, 1] \to [0, 1]$.

Definition 1.1.6. A fuzzy union (t-**conorm**) is a binary relation u on the unit interval that satisfies the following properties: $\forall a, b, d \in [0, 1]$:

(1) $u(a, 0) = a$ (boundary condition);

(2) $b \leq d$ implies $u(a, b) \leq u(a, d)$ (monotonicity);

(3) $u(a, b) = u(b, a)$ (commutativity);

(4) $u(a, u(b, d)) = u(u(a, b), d)$ (associativity).

Example 1.1.7. Let $a, b \in [0, 1]$.

Standard union: $u(a, b) = a \vee b$;

Algebraic sum: $u(a, b) = a + b - ab$;

Bounded sum: $u(a, b) = 1 \wedge (a + b)$.

A special kind of aggregation operations are operations h on [0, 1] that satisfy the properties of monotonicity, commutativity, and associativity, but replace the boundary conditions of t-norms and t-conorms with weaker boundary conditions:

$$h(0, 0) = 0 \text{ and } h(1, 1) = 1.$$

These operations are called **norm operations**.

Example 1.1.8. Let i be a t-norm and u be a t-conorm. Let $\lambda \in [0, 1]$. Let h_λ be the fuzzy binary relation on [0, 1] defined by for all $a, b \in [0, 1]$,

$$h_\lambda(a, b) = \begin{cases} \lambda \wedge u(a, b) \text{ if } a, b \in [0, \lambda], \\ \lambda \vee i(a, b) \text{ if } a, b \in [\lambda, 1], \\ \lambda \ \text{ otherwise.} \end{cases}$$

Then h_λ is a norm operation.

1.2 Evidence theory

Evidence theory is one of the broadest frameworks for the representation of uncertainty. Its origins lie in the works of Dempster [27], [28] and Shafer [102] and are heavily influenced by probability theory, one of the oldest uncertainty frameworks. Evidence theory encompasses belief, plausibility, necessity, possibility, and probability among a host of other measures. Here we present Evidence theory as it was originally characterized by Shafer.

Evidence theory is based on two fuzzy measures: belief measures and plausibility measures. Belief and plausibility measures can be conveniently characterized by a function m from the power set of a universal set X into the unit interval. We assume that X is finite in this section. The function $m : \mathcal{P}(X) \to [0, 1]$ is required to satisfy two conditions:

(1) $m(\emptyset) = 0$;

(2) $\sum_{A \in \mathcal{P}(X)} m(A) = 1$.

The function m is called a **basic probability assignment**. For each set $A \in \mathcal{P}(X)$, the value $m(A)$ expresses the proportion to which all available and relevant evidence supports the claim that a particular element of X belongs to the set A. This value $m(A)$ pertains solely to one set, set A; it does not imply any additional claims regarding subsets or supersets of A. If there is some additional subset of A, say $B \subseteq A$, it must be expressed by another value $m(B)$.

Given a basic probability assignment, m, every set $A \in \mathcal{P}(X)$ for which $m(A) \neq 0$ is called a **focal element**. The pair $(\mathcal{F}, m)$, where $\mathcal{F}$ denotes the set of all focal elements induced by m is called a **body of evidence** and we denote it by $\mathcal{B} = (\mathcal{F}, m)$.

From a basic probability assignment m, the corresponding belief measure and plausibility measure are determined for all sets $A \in \mathcal{P}(X)$ by the formulas,

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B),$$

$$\mathrm{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(B).$$

Thus the belief of a set A is the sum of all the evidence (basic probability) assigned to A or any subset of A. The plausibility of A is the sum of all the evidence (basic probability) that overlaps with A.

It can be shown that the plausibility of an event is one minus the belief of the compliment of that event, and vice versa. That is,

$$\mathrm{Bel}(A) = 1 - \mathrm{Pl}(A^c),$$

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(A^c).$$

Since we can calculate the belief from the plausibility, and the plausibility from the belief, and both belief and plausibility can be derived from the basic probability assignment, we only need one formula to show that all three measures provide the same information.

Given a belief measure Bel, the corresponding basic probability assignment m is determined for all $A \in \mathcal{P}(X)$ by the formula

$$m(A) = \sum_{B \subset A} (-1)^{|A-B|} \mathrm{Bel}(B),$$

where $|A - B|$ is the cardinality of the set difference of A and B, as proven by Shafer [102]. Thus each of the three functions, m, Bel, and Pl is sufficient to determine the other two.

Total ignorance is expressed in evidence theory by $m(X) = 1$ and $m(A) = 0$ for all $A \subset X$. Full certainty is expressed by $m(\{x\}) = 1$ for one particular element x of X and $m(A) = 0$ for all $A \neq \{x\}$.

Guiasu method

The Guiasu method describes the process of reaching a verdict by probabilistic weighting the available evidence. The classical rules from decision theory proposed by Hooper, Dempster, Bayes, and Jeffrey are special cases of Guiasu's weighting process. The Guiasu method is a generalization of Dempster-Shafer theory [27,28, 102] and makes use of fuzzy set theory.

A body of information induces a probability (credibility) distribution m on $\mathcal{P}(X)$, the set of all subsets of X. That is, m is a function of $\mathcal{P}(X)$ into the closed interval $[0, 1]$, written $m : \mathcal{P}(X) \to [0, 1]$, such that $m(A) \geq 0 \forall A \in \mathcal{P}(X)$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$. The class of focal subsets of X corresponding to m is denoted by $\mathcal{F}(X; m) = \{A | A \subseteq X, m(A) > 0\}$. A pair of dependent bodies of information, say i and j, induce a joint probability (credibility) distribution, namely $m_{ij} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$ such that $m_{ij}(A, B) \geq 0$ and $\sum_{A \subseteq X} m_{ij}(A, B) = 1$. If the bodies of information are independent, then $m_{ij} = m_i m_j$. The corresponding class of focal pairs of subsets is $\mathcal{F}(X, X; m_{ij}) = \{(A, B) | A, B \subseteq X, m_{ij}(A, B) > 0\}$. The weights corresponding to the body of information for which m is the probability (credibility) distribution on $\mathcal{P}(X)$ are $w(\cdot|\cdot) : \mathcal{P}(X) \times \mathcal{F}(X; m) \to [0, \infty)$. The weighted body of information provides the new probability (credibility) distribution on $\mathcal{P}(X)$ given by $\mu(C) = \sum_{A \in \mathcal{F}(X;m)} w(C|A)m(A)$. We can generalize this procedure to formulate the weights $w_{ij}(\cdot|\cdot, \cdot)$ that are assigned to a mixed body of information inducing a joint probability (credibility) distribution induced on $\mathcal{P}(X)$ by the weighted (i, j)-th body of information, i.e.,

$$\mu_{ij}(C) = \sum_{(A,B) \in \mathcal{F}(X, X m_{ij})} w_{ij}(C|A, B) m_{ij}(A, B), C \in \mathcal{P}(X),$$

where $w_{ij}(C|A, B)$ is the weight of the subset C given $(A, B) \in \mathcal{F}(X, X; m_{ij})$. If the probability (credibility) distribution m on X is such that $\sum_{A \in \mathcal{F}(X;m)} m(A) = 1$ and $\forall A \in \mathcal{F}(X; m), |A| = 1$, then it is called **probabilistic**.

The following discussion is explained via sustainable development goals (SDGs). Given m subgoals (SDGs in this application) and n experts. Assume the experts assign numbers to each SDG a number with respect to their importance in the examination of the overarching goal (sustainability) to form a $m \times n$ matrix $W = [w_{ij}]$. When the columns of the matrix are normalized, we can consider that each column of the resulting matrix N to be a probability (credibility) distribution for each expert (t-norm in this application). These probability (credibility) distributions are probabilistic with the focal elements being singleton sets consisting of an SDG. The row averages provide the Guiasu weights, one for each SDG.

Theorem 1.2.1. *[84] The row averages of N give the Guiasu weights $w_i, i = 1, ..., m$.*

Analytic hierarchy process

The analytical hierarchy process (AHP) is a multicriteria decision method introduced in [93] and [94]. We consider a factor to be studied by the examination of subfactors of the factor. In our case, each expert E_j assigns a number, w_{ij}, to each subfactor (SDG), $G_i, i = 1, ..., m$, as to its importance with respect to the overarching goal (sustainability). The row average, w_i, of each row of the matrix $W = [w_{ij}]$ is determined to form an $m \times n$-matrix R whose ij-th element is w_i/w_j. The columns of R are then normalized in order to form the $m \times n$-matrix N whose ij-th element is $(w_i/w_j)/\sum_{i=1}^{m} w_i/w_j = w_i/\sum_{i=1}^{m} w_i$, $i = 1, ..., m$. This row vector yields the weights for the subfactors (SDGs) for the linear equation of the overarching goal (sustainability), the dependent variable, in terms of the SDGs, the independent variables.

If the matrix W already has its columns normalized, then $w_i = \sum_{j=1}^{n} w_{ij}/n$, $i = 1, ..., m$. Since $\sum_{i=1}^{m} w_{ij} = 1$, $j = 1, ..., n$, it follows that $\sum_{i=1}^{m} w_i = 1$. Hence $w_i/\sum_{i=1}^{m} w_i = w_i$, i.e., w_i is the weight for the i-th SDG in the linear equation, $i = 1, ..., m$. It thus follows that if the columns of W are already normal, then the Guiasu method (with probabilistic assignments) and the analytic hierarchy process yield the same weights. However, in general, the Guiasu weights and the analytic hierarchy process can have quite different weights [84].

Yen method

Yen's method addresses the issue of managing imprecise and vague information in evidential reasoning by combining the Dempster-Shafer theory with fuzzy set theory [127]. Several researchers have extended the Dempster-Shafer theory to deal with vague information, but their extensions did not preserve an important principle that the belief and plausibility measures are lower and upper probabilities. Yen's method preserves this principle. Nevertheless, we use various measures of subsethood to determine belief functions. We do this to compare the results of the beliefs with Yen's method.

Yen's method is developed under the assumption that the focal elements are normal. If the fuzzy focal elements are not normal, he normalizes them.

1.3 Fuzzy similarity measures

In this section, we take a quick look at fuzzy similarity measures. We are interested in measuring the similarity between rankings.

Definition 1.3.1. Let S be a function of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ into $[0, 1]$. Then S is called a **fuzzy similarity measure** on $\mathcal{FP}(X)$ if the following properties hold: $\forall \mu, \nu, \rho \in \mathcal{FP}(X)$,

(1) $S(\mu, \nu) = S(\nu, \mu)$;

(2) $S(\mu, \nu) = 1$ if and only if $\mu = \nu$;

(3) If $\mu \subseteq \nu \subseteq \rho$, then $S(\mu, \rho) \leq S(\mu, \nu) \wedge S(\nu, \rho)$;

(4) If $S(\mu, \nu) = 0$, then $\forall x \in X$, $\mu(x) \wedge \nu(x) = 0$.

In this section, we consider similarity measures. We apply them in a new way. Suppose that X is a finite set with n elements. Let A be a one-to-one function of X onto $\{1, 2, ..., n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X as follows: $\forall x \in X$, $\mu_A(x) = A(x)/n$. We wish to consider the similarity of two rankings of X by the use of similarity measures.

Much of our discussion is from [122].

Example 1.3.2. Let μ_A and μ_B be the fuzzy subsets of X associated with two rankings A and B of X, respectively. Then M, L, and S are similarity measures.

$$M(\mu_A, \mu_B) = \frac{\sum_{x \in X} \mu_A(x) \wedge \mu_B(x)}{\sum_{x \in X} \mu_A(x) \vee \mu_B(x)};$$
$$L(\mu_A, \mu_B) = 1 - \vee_{x \in X} |\mu_A(x) - \mu_B(x)|;$$
$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x \in X} |\mu_A(x) - \mu_B(x)|}{\sum_{x \in X} (\mu_A(x) + \mu_B(x))}.$$

Lemma 1.3.3. *Let $a, b, c \in [0, 1]$. Then $(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c$.*

Lemma 1.3.4. *Let $x, y, z \in (0, 1]$ be such that $x \leq y$. Then $(x \vee z)/(y \vee z) \geq x/y$.*

Proof. The result follows by considering the three cases $z \leq x \leq y$, $x \leq z \leq y$, and $x \leq y \leq z$. ■

Theorem 1.3.5. $M(\mu_A \cup \mu_C, \mu_B \cup \mu_C) \geq M(\mu_A, \mu_B)$.

Proof. Applying the previous lemmas, we have that

$$\begin{aligned} M(\mu_A \cup \mu_C, \mu_B \cup \mu_C) &= \frac{\sum_{x \in X} (\mu_A(x) \vee \mu_C(x)) \wedge (\mu(_B(x) \vee \mu_C(x))}{\sum_{x \in X} \mu_A(x) \vee \mu_C(x) \vee \mu_B(x) \vee \mu_C(x)} \\ &= \frac{\sum_{x \in X} (\mu_A(x) \wedge \mu_B(x)) \vee \mu_C(x))}{\sum_{x \in X} \mu_A(x) \vee \mu_B(x) \vee \mu_C(x)} \\ &\geq \frac{\sum_{x \in X} \mu_A(x) \wedge \mu_B(x)}{\sum_{x \in X} \mu_A(x) \vee \mu_B(x)} \\ &= M(\mu_A, \mu_B). \end{aligned}$$ ■

Lemma 1.3.6. *Let $x, y, z \in [0, 1]$. Then $|(x \vee z) - (y \vee z)|/((x \vee z) + (y \vee z)) \leq |x - y|/(x + y)$.*

Proof. There is no loss in generality in assuming $x \geq y$. Suppose $z \geq x \geq y$. Then $(x \vee z) - (y \vee z) = z - z \leq x - y$. Assume $x \geq y \geq z$. Then $(x \vee z) - (y \vee z) = x - y$. Suppose $x \geq z \geq y$. Then

$$\begin{aligned} y &\leq z \\ 2xy &\leq 2xz \\ xy - zx &\leq -xy + zx \\ x^2 + xy - zx - zy &\leq x^2 - xy + zx - zy \\ (x - z)(x + y) &\leq (x + z)(x - y) \\ \frac{x - z}{x + z} &\leq \frac{x - y}{x + y} \\ \frac{x \vee z - y \vee z}{x \vee z + y \vee z} &\leq \frac{x - y}{x + y}. \end{aligned}$$ ■

Theorem 1.3.7. $S(\mu_A \cup \mu_C, \mu_B \cup \mu_C) \geq S(\mu_A, \mu_B)$.

Proof. We have by Lemma 1.3.6 that

$$\begin{aligned} S(\mu_A \cup \mu_C, \mu_B \cup \mu_C) &= 1 - \frac{\sum_{x\in X} |(\mu_A(x) \vee \mu_C(x)) - (\mu(_B(x) \vee \mu_C(x))|}{\sum_{x\in X} \mu_A(x) \vee \mu_C(x) + \mu_B(x) \vee \mu_C(x)} \\ &\geq 1 - \frac{\sum_{x\in X} |\mu_A(x) - \mu(_B(x)|}{\sum_{x\in X} \mu_A(x) + \mu_B(x)} \\ &= S(\mu_A, \mu_B). \end{aligned}$$

■

Lemma 1.3.8. *Let $a, b, c \in [0, 1]$ be such that $a \geq b$. Then*

(1) $a - b \geq a \vee c - b \vee c$;

(2) $a - b \geq a \wedge c - b \wedge c$.

Proof. (1) Suppose $(i)c \geq a \geq b$. Then $a \vee c - b \vee c = c - c \leq a - b$. Suppose $(ii)a \geq c \geq b$. Then $a \vee c - b \vee c = a - c \leq a - b$. Suppose $(iii)a \geq b \geq c$. Then $a \vee c - b \vee c = a - b$.

(2) Suppose $(i)c \geq a \geq b$. Then $a \wedge c - b \wedge c = a - b$. Suppose $(ii)a \geq c \geq b$. Then $a \wedge c - b \wedge c = c - b \leq a - b$. Suppose $(iii)a \geq b \geq c$. Then $a \vee c - b \vee c = c - c = 0 \leq a - b$. ■

Theorem 1.3.9. *Let μ_A, μ_B, μ_C be fuzzy subsets of X. Then*

(1) $L(\mu_A, \mu_B) \leq L(\mu_A \cup \mu_C, \mu_B \cup \mu_C)$;

(2) $L(\mu_A, \mu_B) \leq L(\mu_A \cap \mu_C, \mu_B \cap \mu_C)$.

Proof. (1) By (1) of Lemma 1.3.8, we have for all $x \in X$ that

$$\begin{aligned} |\mu_A(x) - \mu_B(x)| &\geq |\mu_A(x) \vee \mu_C(x) - \mu_B(x) \vee \mu_C(x)| \\ 1 - |\mu_A(x) - \mu_B(x)| &\leq 1 - |\mu_A(x) \vee \mu_C(x) - \mu_B(x) \vee \mu_C(x)|. \end{aligned}$$

A similar argument holds for (2). ■

Theorem 1.3.10. *[71, pp. 12-14] Let M and S be the fuzzy similarity measures of Example 1.3.2. Let X be a set with n elements and let A and B rankings of X.*

(1) *If n is even, then the smallest $M(\mu_A, \mu_B)$ can be is $\frac{n+2}{3n+2}$ and the smallest $S(\mu_A, \mu_B)$ be is $\frac{n/2+1}{n+1}$.*

(2) *If n is odd, then the smallest $M(\mu_A, \mu_B)$ can be is $\frac{n+1}{3n-1}$ and the smallest $S(\mu_A, \mu_B)$ be is $\frac{1}{2} + \frac{1}{2n}$.*

Theorem 1.3.11. *[73] Let M and S be the fuzzy similarity measures s in Theorem 1.3.10. Then $S = \frac{2M}{1+M}$.*

1.4 Implication operations and similarity operations

In this section, we define and examine similarity measures in terms of implication operators. Our discussion is from [9,122].

Definition 1.4.1. Let $\mathcal{I}$ be a function of $[0, 1] \times [0, 1]$ into $[0, 1]$ such that $\mathcal{I}(0, 0) = \mathcal{I}(0, 1) = \mathcal{I}(1, 1) = 1$ and $\mathcal{I}(1, 0) = 0$. Then $\mathcal{I}$ is called an **implication operation**.

Let μ_A and μ_B be two fuzzy subsets of a set X. Let $\mathcal{I}$ be an implication operator. Then the degree to which μ_A is a subset of μ_B is defined to be

$$\wedge\{\mathcal{I}(\mu_A(x), \mu_B(x))|x \in X\}.$$

Define the fuzzy subset $\subseteq_I$ of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by $\forall \mu_A, \mu_B \in \mathcal{FP}(X)$, $\subseteq_I (\mu_A, \mu_B) = \wedge\{\mathcal{I}(\mu_A(x), \mu_B(x))|x \in X\}$.

Definition 1.4.2. Let $\mathcal{I}$ be an implication operator. Define the fuzzy subset $E_{\mathcal{I}}$ of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by for all $\mu_A, \mu_B \in \mathcal{FP}(X)$,

$$E_{\mathcal{I}}(\mu_A, \mu_B) = \wedge\{\wedge\{\mathcal{I}(\mu_A(x), \mu_B(x))\,|x \in X\}, \wedge\{\mathcal{I}(\mu_B(x), \mu_A(x))|x \in X\}\}.$$

Then is called the **degree of sameness** of μ_A and μ_B [9].

Let $\mathcal{T}$ denote a t-norm. Then there exists an implication operator $\mathcal{I}_{\mathcal{T}}$ defined by $\mathcal{I}_{\mathcal{T}}(x, y) = \vee\{z|z \in [0, 1]$ and $\mathcal{T}(x, z) \leq y\}$. The following implication operators can be determined by a suitable t-norm, [10].

Example 1.4.3. Let $x, y \in [0, 1]$.

(1) Godel implication operator: $\mathcal{I}(x, y) = \begin{cases} 1 \text{ if } x \leq y, \\ y \text{ otherwise.} \end{cases}$

(2) Goguen implication operator: $\mathcal{I}(x, y) = \begin{cases} 1 \text{ if } x \leq y, \\ y/x \text{ otherwise.} \end{cases}$

(3) Luckasiewicz implication operator: $\mathcal{I}(x, y) = \wedge\{1 - x + y, 1\}$.

Definition 1.4.4. Let $\mathcal{I}$ be an implication operator. Then $\mathcal{I}$ is called **hybrid monotonous** if $\mathcal{I}(x, \ )$ is nondecreasing for all $x \in [0, 1]$ and $\mathcal{I}(\ , y)$ is nonincreasing for all $y \in [0, 1]$.

The implication operators in the previous example are hybrid monotonous.

Proposition 1.4.5. *[122] Let A be a finite subset of* $[0, 1]$ *and* $b \in [0, 1]$. *Let* $\mathcal{I}$ *be a hybrid monotonous implication operator. Then*

(1) $\mathcal{I}(\vee\{a|a \in A\}, b) = \wedge\{\mathcal{I}(a, b)|a \in A\}$;

(2) $\mathcal{I}(\wedge\{a|a \in A\}, b) = \vee\{\mathcal{I}(a, b)|a \in A\}$;

(3) $\mathcal{I}(b, \vee\{a|a \in A\}) = \vee\{\mathcal{I}(a, b)|a \in A\}$;

(4) $\mathcal{I}(b, \wedge\{a|a \in A\}) = \wedge\{\mathcal{I}(b, a)|a \in A\}$.

Lemma 1.4.6. *[122] Let $\mathcal{I}$ be the Luckasiewicz implication operator. Let $a, b \in [0, 1]$. Then*

$$\mathcal{I}(a, b) \wedge \mathcal{I}(b, a) = ((1 - a) \wedge (1 - b)) + a \wedge b.$$

Proof. We show that

$$(1 - a + b) \wedge (1 - b + a) \wedge 1$$
$$= ((1 - a) \wedge (1 - b)) + a \wedge b.$$

Suppose that $a \leq b$. Then

$$((1 - a) \wedge (1 - b)) + a \wedge b = 1 - b + a.$$

Clearly, $1 - a + b \geq 1$ and $1 - b + a \leq 1$. Thus $(1 - a + b) \wedge (1 - b + a) \wedge 1 = 1 - b + a$.

The proof of the case for $a \geq b$ is similar. ∎

Proposition 1.4.7. *[9] Let $\mathcal{I}$ be the Luckasiewicz implication operator. Then for all $\mu_A, \mu_B \in \mathcal{FP}(X)$, $E_{\mathcal{I}}(\mu_A, \mu_B) = L(\mu_A, \mu_B)$.*

Proof. We have that

$$L(\mu_A, \mu_B) = 1 - \vee\{|\mu_A(x) - \mu_B(x)| \mid x \in X\}$$
$$= \wedge\{1 - |\mu_A(x) - \mu_B(x)| \mid x \in X\}$$
$$= \wedge\{1 - \mu_A(x) \vee \mu_B(x) + \mu_A(x) \wedge \mu_B(x)) | x \in X\}$$
$$= \wedge\{(1 - \mu_A(x)) \wedge (1 - \mu_B(x)) + \mu_A(x) \wedge \mu_B(x) | x \in X\}.$$

It suffices to show that

$$\mathcal{I}(\mu_A(x), \mu_B(x)) \wedge \mathcal{I}(\mu_B(x), \mu_A(x))$$
$$= (1 - \mu_A(x)) \wedge (1 - \mu_B(x)) + \mu_A(x) \wedge \mu_B(x).$$

However, this holds from Lemma 1.4.6. ∎

We next consider the interactions between the concept of degree of sameness and fuzzy set theoretical operations.

A fuzzy complement c is called **involutive** if for all $x \in [0, 1]$, $c(c(x)) = x$.

An implication operator $\mathcal{I}$ is called **contrapositive** (with respect to a fuzzy complement c) if $\forall x, y \in [0, 1]$, $\mathcal{I}(x, y) = \mathcal{I}(c(y), c(x))$. Note that the standard complement is involutive.

Proposition 1.4.8. *[122] Let $\mathcal{I}$ be a contrapositive implication operator with respect to an involutive fuzzy complement c. Let μ, ν be fuzzy subsets of X. Then $E_{\mathcal{I}}(\mu, \nu) = E_{\mathcal{I}}(\nu^c, \mu^c)$.*

Proof. We have that

$$\begin{aligned} E_{\mathcal{I}}(\mu, \nu) &= (\wedge\{\mathcal{I}(\mu(x), \nu(x))|x \in X\}) \wedge (\wedge\{\mathcal{I}(\nu(x), \mu(x))|x \in X\}) \\ &= (\wedge\{\mathcal{I}(\nu^c(x), \mu^c(x))|x \in X\}) \wedge (\wedge\{\mathcal{I}(\mu^c(x), \nu^c(x))|x \in X\}) \\ &= E_{\mathcal{I}}(\nu^c, \mu^c). \end{aligned}$$

■

Proposition 1.4.8 holds for Kleene-Dienes implication operator, $\mathcal{I}(x, y) = (1 - x) \vee y$ and the Early Zadeh implication operator $\mathcal{I}(x, y) = (x \wedge y) \vee (1 - x)$ even though these implication operators are not contrapositive [122].

1.5 Fuzzy graphs

Let V be a nonempty set. Let $\mathcal{E}$ denote the set of all subsets of V with cardinality 2. Let $E \subseteq \mathcal{E}$. A graph is a pair (V, E). The elements of V are thought of as vertices of the graph and E as the set of edges. For $\{x, y\} \in E$, we let xy denote $\{x, y\}$. Then clearly $xy = yx$.

Definition 1.5.1. [57,92] Let (V, E) be a graph. Then the pair (Ω, Ψ) is called a **fuzzy subgraph** of (V, E) if Ω is a fuzzy subset of V and ψ is a fuzzy subset of E such that for all $xy \in E$, $\Psi(xy) \leq \Omega(x) \wedge \Omega(y)$.

Most of the definitions provided here are from [92].

Definition 1.5.2. Let (Ω, Ψ) be a fuzzy subgraph of the graph (V, E). Then a fuzzy subgraph (τ, ν) of (V, E) is called a **partial fuzzy subgraph** (Ω, Ψ) if $\tau \subseteq \Omega$ and $\nu \subseteq \Psi$.

Definition 1.5.3. Let (Ω, Ψ) be a fuzzy subgraph of the graph (V, E). Then a partial fuzzy subgraph (τ, ν) of (Ω, Ψ) is said to **span** (Ω, Ψ) if $\tau = \Omega$. In this case, (τ, ν) is called a **spanning fuzzy subgraph** of (Ω, Ψ).

A **path** P in a fuzzy graph (Ω, Ψ) of a graph (V, E) is a sequence of distinct vertices $x_0, x_1, ..., x_n$ (except possibly $x_0, ..., x_n$) such that $\Psi(x_{i-1}x_i) > 0$, $i = 1, ..., n$. Here n is called the **length** of the path. The consecutive pairs are called the edges of the path. The **diameter** of $x, y \in V$, written diam(x, y), is the length on the longest path joining x and y. The **strength** of P is defined to be $\wedge_{i=1}^{n} \Psi(x_{i-1}x_i)$. The strength of connectedness between two vertices x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $\Psi^{\infty}(x, y)$ or CONN (x, y). A **strongest path** joining any two vertices x and y has strength $\Psi^{\infty}(x, y)$. It can be shown that if (τ, ν) is a partial fuzzy subgraph of (Ω, Ψ) then $\nu^{\infty} \subseteq \Psi^{\infty}$. We call P a **cycle** if $x_0 = x_n$ and $n \geq 3$. Two vertices that are joined by a path are called **connected**. It follows that this notion of connectedness is an equivalence relation. The equivalence classes of vertices under this equivalence relation are called **connected components** of the given fuzzy subgraph. They are its maximal connected partial fuzzy subgraphs.

Let $G = (\Omega, \psi)$ be a fuzzy graph, let x, y be distinct vertices, and let G' be the partial fuzzy subgraph of G obtained by deleting the edge xy. That is, $G' = (\sigma, \mu')$, where $\mu'(x, y) = 0$ and $\mu' = \mu$ for all other pairs. We call xy a **fuzzy bridge** in G if $\mu'^{\infty}(u, v) < \mu^{\infty}(u, v)$ for some u, v in σ^*. In words, the deletion of xy reduces the strength of connectedness between some pair of vertices in G. Thus xy is a fuzzy bridge if and only if there exists $u, v \in V$ such that xy is an edge of every strongest path between u and v.

Similarly, if the strength of connectedness between some pair of vertices is reduced, when it is removed from the fuzzy graph, then such a vertex is called a **fuzzy cutvertex**. Fuzzy graph with no fuzzy cutvertices is called **nonseparable** or a **block**. If $\mu(ab) = \sigma(a) \wedge \sigma(b)$ for all $a, b \in \Omega*$ then such a fuzzy graph is called a **complete fuzzy graph (CFG)** [11].

Definition 1.5.4. A connected fuzzy graph is called a **fuzzy tree** if it contains a spanning subgraph F which is a tree such that, for all edges ab not in F, $\mu(ab) < CONN_F(a, b)$.

If (Supp(Ω), Supp(ψ)) is a cycle then $G = (\Omega, \psi)$ is called a **cycle**. A fuzzy graph G is a **fuzzy cycle** if G is a cycle and it has more than one edge with minimum weight.

Definition 1.5.5. [14,62] If an edge ab belongs to $Supp(\Psi)$ and $\Psi(ab) \geq CONN_{G\backslash ab}(a, b)$, then such an edge is called a **strong edge.** An edge ab is called *α-strong* if $\Psi(ab) > CONN_{G\backslash ab}(a, b)$, it is called *$\beta$-strong* if $\Psi(ab) = CONN_{G\backslash ab}(a, b)$ and is called a *δ-edge* if $\Psi(ab) < CONN_{G\backslash ab}(a, b)$.

If every edge of a path is α-strong, then the path is said to be an **α-strong path**. Similarly, **β-strong path** is also defined. A strong path P from a to b is called a **geodesic** if there are no shorter strong paths from a to b. If α-strong edges are incident at every vertex of a fuzzy graph, then the graph is called **α-saturated fuzzy graph** [66]. Similarly, **β-saturated fuzzy graph** is also defined. If a fuzzy graph is both α-saturated and β-saturated, then it is called **saturated**, else **unsaturated**.

An **isomorphism** [35] $h : G \to G'$ is a map $h : V \to V'$ which is bijective that satisfies $\Omega(u) = \Omega'(h(u))$ for all $u \in V$, $\Psi(ab) = \Psi'(h(a)h(b))$ for all $a, b \in V$ [12]. The **complement of a fuzzy graph** $G = (\Omega, \Psi)$ is the fuzzy graph $G^c = (\Omega^c, \Psi^c)$ where $\Omega^c = \Omega$ and $\Psi^c(ab) = \Omega(a) \wedge \Omega(b) - \Psi(ab)$ for all $a, b \in V$ [108]. The geodesic eccentricity $l(a)$ of a vertex a is given by $l(a) = \bigvee_{m \in W} d_s(a, m)$ [96]. The sum of weights of all edges incident at a is the degree of a [33]. The sum of weights of all strong edges incident at a is the strong degree of a [63].

Proposition 1.5.6. *[14] Let $G = (\Omega, \Psi)$ be connected, and let a, b be any two vertices in Ω^*. Then there exists a strong path from a to b.*

Theorem 1.5.7. *[63] Let $G = (\Omega, \Psi)$ be a complete fuzzy graph. Then G does not contain any δ-edges.*

Definition 1.5.8. [66] Let $G = (\Omega, \Psi)$ be a fuzzy graph. Let a and b be any two vertices of G. Then the **strong distance** between a and b is defined and denoted by $d_s(a,b) = \wedge\{\sum_{ab\in P_i} \mu(a,b)$: P_i belongs to the set of all strong paths from a to $b\}$, 0 if $a = b$, ∞ if there exists no strong $a - b$ path.

Theorem 1.5.9. *[63] Let $G = (\Omega, \Psi)$ be a CFG with $|\Omega^*| = n$. Then G has a fuzzy bridge if and only if there exists an increasing sequence $\{t_1, t_2, \cdots, t_n\}$ such that $t_{n-2} < t_{n-1} \leq t_n$, where $t_i = \Omega(w_i)$ for $i = 1, 2, \cdots, n$. Also, the edge $w_{n-1}w_n$ is a fuzzy bridge of G.*

Definition 1.5.10. [15] For a fuzzy graph $G = (\sigma, \mu)$, the **Connectivity Index** (CI) is defined as $CI(G) = \sum_{m,p\in\sigma^*} \sigma(m)\sigma(p)CONN_G(m,p)$, where $CONN_G(m,p)$ is the strength of connectedness between m and p. For a fuzzy graph $G = (\sigma, \mu)$, the **Wiener Index** (WI) [17] is defined as $WI(G) = \sum_{m,p\in\sigma^*} \sigma(m)\sigma(p)d_S(m,p)$, where $d_S(m,p)$ is the minimum sum of weights of geodesics from m to p.

1.6 Lattices

Let X be a set and $\leq$ a relation on X. Then $\leq$ is called a partial order on X if $\leq$ is reflexive, antisymmetric, and transitive. Let $\vee, \wedge$ be commutative, associative operations on X such that for all $x, y \in X$

$$x \wedge (x \vee y) = x,$$
$$x \vee (x \wedge y) = x.$$

If we define the binary relation $\leq$ on X by $x \leq y$ if and only if $x \wedge y = x$, then $\leq$ is a partial order on X. Then $L = (X, \leq)$ is called a **lattice** if for all $x, y \in X$, $x \vee y \in X$ and $x \wedge y \in X$. Let $L_i = (X_i, \leq_i)$ be lattices on X_i, $i = 1, ..., n$. Let $X = X_1 \times ... \times X_n$ denote the Cartesian product of the X_i, $i = 1, ..., n$. Define $\leq$ on X by for all $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, $x \leq y$ if and only if $x_y \leq_i y_i$ for all $i = 1, ..., n$. Then $L = (X, \leq)$ is a lattice.

For two partially ordered sets, $(X_1, \leq_1)$ and $(X_2, \leq_2)$, a function $f : X_1 \to X_2$ is called an **order homomorphism** if $x \leq_{X_1} y$ implies $f(x) \leq_{X_2} f(y)$ for all $x, y \epsilon X_1$. If $L_1 = (X_1, \leq_1)$ and $L_2 = (X_2, \leq_2)$ are lattices, a function $f : X_1 \to X_2$ is called a **lattice homomorphism** if for all $x, y \epsilon X_1$,

$$f(x \wedge_{L_1} y) = f(x) \wedge_{L_2} f(y),$$
$$f(x \vee_{L_1} y) = f(x) \vee_{L_2} f(y).$$

A one-to-one homomorphism is called an **isomorphism**.

1.7 Exercises

1. Let $X = \{a, b, c, d\}$. Let μ and ν be fuzzy subsets of X defined as follows:

$$\mu(a) = 0.3,\ \mu(b) = 0.2,\ \mu(c) = 0.5,\ \mu(d) = 0.7,$$
$$\nu(a) = 0.8,\ \nu(b) = 0.3,\ \nu(c) = 0.4,\ \nu(d) = 0.1.$$

Find $\mu \cap \nu$, $\mu \cup \nu$, μ^c, and ν^c.

2. Let μ, ν be fuzzy subsets of a set X. Prove that $(\mu \cup \nu)^c = \mu^c \cap \nu^c$ and $(\mu \cap \nu)^c = \mu^c \cup \nu^c$.

3. Let A and B be rankings of a set X. Suppose X has n elements. Prove that $\sum_{x \epsilon X}(A(x) + B(x)) = n(n+1)$.

4. Let A and B be as in Exercise 3. Prove that $\sum_{x \epsilon X}(A(x) \wedge B(x)) + \sum_{x \epsilon X}(A(x) \vee B(x)) = n(n+1)$.

CHAPTER

Lattice isomorphisms, trafficking, and global challenges[*]

2

In this chapter, we will discuss how theoretical results from one family of fuzzy sets can be carried over immediately to another family of fuzzy sets by the use of lattice isomorphisms. We will also show that these families can occur naturally and that applications may not necessarily be carried over using these isomorphisms. We illustrate this using techniques from the study of human trafficking and its analysis using mathematics of uncertainty. Mathematics of uncertainty is a very appropriate tool to use in the study of trafficking. This is because accurate data concerning trafficking in persons is impossible to obtain. The goal of the trafficker is to be undetected. The size of the problem also makes it very difficult to obtain accurate data. Victims are reluctant to report crimes or testify for fear of reprisals, disincentives, both structural and legal, for law enforcement to act against traffickers, a lack of harmony among existing data sources, and an unwillingness of some countries and agencies to share data.

2.1 Fuzzy sets and lattice isomorphisms

One of the most important papers concerning fuzzy set theory in recent years is one by Klement and Mesiar, [56]. In this paper, it is shown that differently defined families of fuzzy sets have lattice structures that are actually isomorphic and so theoretical results for one family can be carried over to another family.

We show by using a real-world problem with real-world data that even though theoretical results can be obtained for one family from another, the two families may arise naturally in an application.

We use the concepts of vulnerability and government response to modern slavery to illustrate our findings. In [118], it is stated that the departing point is the fact that not only fuzzy sets originate in Language, but that they are just "linguistic entities" genetically different from the concept of "crisp sets" whose origin is either in a physical collection of objects, or in a list of them. A new definition of a fuzzy

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00010-6

set is presented by means of two magnitudes: A qualitative one, called a graph, the basic magnitude, and a quantitative one, a scalar magnitude. If the first reflects the language's relational ground of the fuzzy set, the second reflects the (numerical) extensional state in which it currently appears.

We next illustrate these ideas using the concepts of vulnerability and government response with respect to modern slavery, [41].

Vulnerability measures

(1) Government issues

(2) Nourishment and access

(3) Inequality

(4) Disenfranchised groups

(5) Effect of conflicts

Countries are scored with respect to these five measures. Then a weighted average of these scores is taken to provide a single score for each number. For example, the final score for Brazil is 36.4. The countries are placed into regions. Brazil is in the Americas. For this region, the highest score was 69.6 and the smallest was 10.2. The country scores were normalized using the formula (number – minimum)/(maximum – minimum) to obtain $(36.4 - 10.2)/(69.6 - 10.2) = 0.443$

Government response

(1) Support for survivors

(2) Criminal justice

(3) Coordination

(4) Response

(5) Supply chains

Similarly, as for the vulnerability measures a final score is determined for each country. For example, the final score for Brazil is 55.6. For the Americas, the maximum score was 71.7 and the minimum was 20.8. Hence the normalized value for Brazil was $(55.6 - 20.8)/(71.6 - 20.8) = 0.684$.

2.2 New view of fuzzy subsets

In [118], a new definition of a fuzzy set is presented by means of two magnitudes: A quantitative one, the basic magnitude, and a quantitative one. The first reflects the language's relational ground of a fuzzy set, the second reflects the (numerical) extensional state in which it currently appears. To know how a predicate P acts on X, that is, how the elements in X can be distinguished by how they verify P, or how the property P varies along the universe of discourse, it should be known when, given two elements x, y in X, which one of them shows P less than the other. Shortening the statement x less P than y by $x \preceq_P y$. Two fuzzy sets with respective linguistic labels P and Q are coincidental, provided both are in the same universe of discourse and are primarily used in the same form. That is,

$$P = Q \Leftrightarrow X = Y \text{ and } \preceq_P = \preceq_Q .$$

The equality of two fuzzy sets means that their linguistic labels do have the same primary use or meaning.

A function $m_P : X \to [0, 1]$ is called a **measure** of P in X if it satisfies the following three properties:

(i) $x \preceq_P y \Rightarrow m_P(x) \leq m_P(y)$;
(ii) If z is minimal, i.e., there is no w such that $w \prec_P z$, then $m_P(z) = 0$;
(iii) If w is maximal, i.e., there is no z in X such that $w \prec_P z$, then $m_P(w) = 1$.

We show in the following how this approach fits naturally with our examination of modern slavery. We demonstrate this by considering situations where accurate data is not available.

In [118], it is stated that shortening the statement x is less P, where P is a predicate, by $x \prec_P y$ facilitates the basic magnitude. That is, $x \prec_P y \subseteq X \times X$.

For our illustration, we let P denote the predicate vulnerable and X denote the set of countries under consideration. Now the final vulnerable score for Mexico was 57.3. Brazil's was 36.4. Hence Brazil $\prec_P$ Mexico. The final value for government response for Mexico was 52.4 and for Brazil 55.6. In this case, we have Mexico $\prec_P$ Brazil if P denotes government response and $\prec_P$ is the linguistic relation x has less government response than y.

We see that our membership function $m_P(x) = \frac{\#(x) - \min}{\max - \min}$, where $\#(x)$ denotes the final score of x, satisfies these three properties. Thus $m_V(Brazil) = 0.443$, where V denotes vulnerable and $m_G(Brazil) = 0.684$, where G denotes government response. For Mexico, we have $m_V(Mexico) = 0.796$ and $m_G(Mexico) = 0.621$.

We present some isomorphisms and other methods in fuzzy set theory to obtain results from one family for another.

Neutrosophic fuzzy sets and Pythagorean fuzzy sets: Recall that a neutrosophic fuzzy set is a triple (σ, τ, μ) of fuzzy subsets of a set. It is based on the lattice of elements $(x_1, x_2, x_3) \in [0, 1]^3$, where $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$ if and only if $x_1 \leq y_1$, $x_2 \leq y_2$, and $x_3 \geq y_3$, [104]. Also, a Pythagorean fuzzy set is a pair of fuzzy subsets (σ, τ) of a set X such that for all $x \in X$, $\sigma(x)^2 + \tau(x)^2 \leq 1$, [125]. We can see that vulnerability and government response corresponding to modern slavery are opposites, [116]. That is, an increase in government response by a country would lower the country's vulnerability. However, $m_V(Brazil) + m_G(Brazil) = 0.443 + 0.684 > 1$. This gives meaning to neutrosophic fuzzy sets, [104], even though certain theoretical results can follow immediately from other types of fuzzy sets. Also, $(0.443)^2 + (0.684)^2 = 0.096 + 0.468 < 1$. Consequently, similar comments might be able to be made here even though Pythagorean fuzzy sets and intuitionistic fuzzy sets, [8], have corresponding isomorphic lattices. However, this isomorphism may make the situation different to the neutrosophic case since it is so straightforward. The lattice isomorphism f involved here is $f : P^* \to L^*$ defined by $f((x_1, x_2)) = (x_1^2, x_2^2)$, where

$$L^* = \{(x_1, x_2) \in [0, 1] | x_1 + x_2 \le 1\},$$
$$P^* = \{(x_1, x_2) \in [0, 1] | x_1^2 + x_2^2 \le 1\}.$$

The paper by Klement and Mesiar [56] contains many other cases, where various families of fuzzy sets have isomorphic lattices.

Let X be a set with n elements, say $X = \{x_1, ..., x_n\}$. Let μ, ν be fuzzy subsets of X. Consider the fuzzy similarity measures,

$$M(\mu, \nu) = \frac{\sum_{x \in X} \mu(x) \wedge \nu(x)}{\sum_{x \in X} \mu(x) \vee \nu(x)},$$
$$S(\mu, \nu) = 1 - \frac{\sum_{x \in X} |\mu(x) - \nu(x)|}{\sum_{x \in X} (\mu(x) + \nu(x))}.$$

Let m be a positive real number. Then $\frac{\sum_{x \in X} \mu(x) \wedge \nu(x)}{\sum_{x \in X} \mu(x) \vee \nu(x)} = \frac{\sum_{x \in X} \mu(x)/m \wedge \nu(x)/m}{\sum_{x \in X} \mu(x)/m \vee \nu(x)/m}$ and $\frac{\sum_{x \in X} |\mu(x) - \nu(x)|}{\sum_{x \in X} (\mu(x) + \nu(x))} = \frac{\sum_{x \in X} |\mu(x)/m - \nu(x)/m|}{\sum_{x \in X} (\mu(x)/m + \nu(x)/m)}$. Suppose there exists $x \in X$ such that $\mu(x) + \nu(x) > 1$. Let m denote the maximal such $\mu(x) + \nu(x)$. Then we see that we get the same M and S values if we divide all the $\mu(x)$ and $\nu(x)$ by m.

Example 2.2.1. Let $X = \{x_1, x_2\}$. Define the fuzzy subsets μ, ν of X as follows:

	μ	ν
x_1	0.1	0.1
x_2	0.2	0.91

Then $\mu(x_2) + \nu(x_2) = 0.2 + 0.91 = 1.11 > 1$. Now $M(\mu, \nu) = \frac{0.1 \wedge 0.1 + 0.2 \wedge 0.91}{0.1 \vee 0.1 + 0.2 \vee 0.91} = \frac{0.3}{1.01}$.

Define the fuzzy subsets μ', ν' of X as follows:

	μ'	ν'
x_1	0.1	0.2
x_2	0.2	0.82

Then $\mu'(x_2) + \nu'(x_2) = 0.2 + 0.82 = 1.02 > 1$. Now $M(\mu', \nu') = \frac{0.1 \wedge 0.2 + 0.2 \wedge 0.82}{0.1 \vee 0.2 + 0.2 \vee 0.82} = \frac{0.3}{1.02}$.

Thus $M(\mu, \nu) > M(\mu', \nu')$. We have a Pythagorean situation since $(0.2)^2 + (0.91)^2 < 1$ and $(0.2)^2 + (0.82)^2 < 1$.

Squaring the values of μ and ν, we obtain $\mu_2(x_1) = 0.01$, $\mu_2(x_2) = 0.04$, and $\nu_2(x_1) = 0.01$, $\nu_2(x_2) = 0.8281$. Hence

$$M(\mu_2, \nu_2) = \frac{0.01 \wedge 0.01 + 0.4 \wedge 0.8281}{0.01 \vee 0.01 + 0.4 \vee 0.8281} = \frac{0.01 + 0.04}{0.01 + 0.8281}.$$

Also, $\mu_2'(x_1) = 0.01$, $\mu_2'(x_2) = 0.04$, $\nu_2'(x_1) = 0.04$, $\nu_2'(x_2) = 0.6724$. Thus

$$M(\mu_2', \nu_2') = \frac{0.01 \wedge 0.04 + 0.04 \wedge 0.6724}{0.01 \vee 0.04 + 0.04 \vee 0.6724} = \frac{0.01 + 0.04}{0.04 + 0.6724}.$$

Hence $M(\mu_2, \nu_2) < M(\mu_2', \nu_2')$. That is, the inequalities have switched. They were not preserved.

We next consider $S(\mu, \nu) = 1 - \frac{\sum_{x \in X} |\mu(x) - \nu(x)|}{\sum_{x \in X} (\mu(x) + \nu(x))}$. For the previous situation, we have $S(\mu, \nu) = 1 - \frac{0+0.71}{0.2+1.11} = \frac{0.71}{1.31} = 1 - 0.542$ and $S(\mu', \nu') = 1 - \frac{0.1+0.62}{0.3+1.02} = 1 - \frac{0.72}{1.32} = 1 - 0.545$. Thus $S(\mu, \nu) > S(\mu', \nu')$.

We also have $S(\mu_2, \nu_2) = 1 - \frac{0+0.7881}{0.04+0.8281} = 1 - \frac{0.7881}{0.8681} = 1 - 0.9078$ and $S(\mu_2', \nu_2') = 1 - \frac{0.03+.6324}{0.05+0.7164} = 1 - \frac{0.6624}{0.7664} = 1 - 0.8643$. Hence $S(\mu_2, \nu_2) < S(\mu_2', \nu_2')$. Once again the inequalities were not preserved.

We have shown with this example that the isomorphism $f : P^* \to L^*$ defined by $f((x_1, x_2)) = (x_1^2, x_2^2)$, where $L^* = \{(x_1, x_2) \in [0, 1] | x_1 + x_2 \leq 1\}$ and $P^* = \{(x_1, x_2) \in [0, 1] | x_1^2 + x_2^2 \leq 1\}$ shows that although theoretical results can determined between Pythagorean fuzzy sets and intuitionistic fuzzy sets, the isomorphism may not be suitable in changing a data set from one to another in applications. Also, isomorphisms in general preserve certain structural properties, but not all outside functions defined on the sets.

We show in the following how this approaches fits naturally with our examination of modern slavery. We demonstrate this by considering situations where accurate data is not available.

Linguistic variables The size of flow of trafficked people from country to country is given in [115]. It is reported in linguistic terms since accurate data concerning the size of the flow is impossible to obtain. Information is provided with respect to the reported human trafficking in terms of origin, transit, and/or destination according to the citation index. The data is provided in two columns. Information in the left column as to whether a country ranks (very) low, medium (very) high depends upon the total number of sources which made reference to this country as one of origins, transit, or destination. Information provided in then the right column provides further detail to the information provided in the left column. If a country in the right column was mentioned by one or two sources, the related country was ranked low. If linkage between the countries in the two columns was reported by 3-5 sources, the related country was ranked medium. If 5 or more sources linked the two countries, the country in the right was ranked high. This method of combining linguistic data provides an ideal reason for the use of mathematics of uncertainty to study the problem of trafficking by persons. For example, by assigning numbers in the interval $[0, 1]$ to the linguistic data, the data can be combined in a mathematical way. In [95], the notions of t-norms and t-conorms were used. The number 0.1 can be assigned very low, 0.3 to low. 0.5 to medium, 0.7 to high, and 0.9 to very high. Using the notation and ideas from [117], we have $x \prec_P y$ if and only if country x's linguistic rank is less than country y's linguistic rank. We have $m_P(x) = 0.1, 0.3, 0.5, 0.7$, or 0.9 if x is assigned very low, low, medium, high, or very high, respectively. We note that here m_P does not satisfy (ii) and (iii).

Colors

In [99], colors are used to determine how well a country is achieving the Sustainable Development Goals (SDGs). A green rating on the SDG dashboard is assigned to a country if all the indicators under that goal are labeled green. Yellow, orange, and red indicate increasing distance from the SDG achievement. The worst two colors of a target were averaged to determine the color of its SDG. In [72], the numbers 0.2, 0.4, 0.6, and 0.8 are assigned to the colors red, orange, yellow, and green, respectively. Consequently, the results in [99] are placed into the context of mathematics of uncertainty.

***t*-norms and *t*-conorms** Suppose X denotes a set of countries involved in human trafficking. Suppose also that $x, y \in X$ represent vertices of a graph and suppose that xy is an edge in the graph such that there is trafficking between x and y. Then $m_V(x)$ and $m_V(y)$ denotes the measure of vulnerability for x and y, respectively. Suppose $m_V(x) = 0.6$ and $m_V(y) = 0.8$. If we wish to determine a joint vulnerability, we might use a t-conorm, say maximum. Then $m_V(x) \vee m_V(y) = 0.8$. However if $m_V(x) = 0.1$, then $m_V(x) \vee m_V(y) = 0.8$ also. It seems that the latter result should be smaller since $0.1 < 0.6$. Thus it seems more realistic to use another t-conorm, say algebraic sum $\oplus$. Then $m_V(x) \oplus m_V(y) = 0.82$ (compared with 0.92). Now consider $m_G(x)$ and $m_G(y)$ as the measure of government response for x and y, respectively. Suppose that $m_G(x) = 0.7$ and $m_G(y) = 0.4$. If we wish to determine a joint government response, we might use a t-norm, say minimum. Then $m_G(x) \wedge m_G(y) = 0.4$. However if $m_G(x) = 0.5$, then $m_G(x) \wedge m_G(y) = 0.4$ also. It seems that the latter result should be smaller since $0.5 < 0.7$. Hence it seems more realistic to use a different t-norm, say product $\bullet$. Then $m_G(x) \bullet m_G(y) = 0.2$ (compared with 0.28).

2.3 Global challenges

There are many global challenges facing the world today. They include global poverty, global hunger, human trafficking, modern slavery, immigration, homelessness, terrorism, biodiversity, extinction, and pandemics. The worst challenge is climate change. Climate change creates poverty which makes all the other challenges worse. Also, climate change can make the planet uninhabitable.

We first concentrate on human trafficking and its analysis using mathematics of uncertainty. Mathematics of uncertainty is a very appropriate tool to use in the study of human trafficking. As previously stated, this is because accurate data concerning trafficking in persons is impossible to obtain. The goal of the trafficker is to be undetected. The size of the problem also makes it very difficult to obtain accurate data. Victims are reluctant to report crimes or testify for fear of reprisals, disincentives, both structural and legal, for law enforcement to act against traffickers, a lack of harmony among existing data sources, and an unwillingness of some countries and agencies to share data.

Table 2.1 is from [41].

Table 2.1 Americas.

Country	Government Response	Vulnerability	Prevalence
Argentina	0.821	0.297	0.156
Barbados	0.365	0.533	0.431
Bolivia	0.402	0.570	0.313
Brazil	0.684	0.441	0.294
Canada	0.742	0.000	0.000
Chile	0.815	0.259	0.058
Columbia	0.398	0.696	0.431
Costa Rica	0.573	0.306	0.156
Cuba	0.000	0.710	0.647
Dominican Rep.	0.730	0.553	0.686
El Salvador	0.326	0.681	0.392
Ecuador	0.502	0.523	0.372
Guatemala	0.479	0.705	0.470
Guyana	0.210	0.592	0.509
Haiti	0.371	1.000	1.000
Honduras	0.318	0.762	0.568
Jamaica	0.742	0.572	0.411
Mexico	0.620	0.792	0.431
Nicaragua	0.500	0.567	0.470
Paraguay	0.394	0.516	0.215
Panama	0.453	0.441	0.313
Peru	0.622	0.574	0.411
Suriname	0.123	0.537	0.352
Trinidad and Tobago	0.571	0.486	0.490
United States	1.000	0.095	0.156
Uruguay	0.581	0.159	0.098
Venezuela	0.145	0.803	1.000

Let m and n be real numbers that are ≥ 1. Let $P_{m,n} = \{(x, y)|x, y \in [0, 1]$ and $x^m + y^n \leq 1\}$. Then $L^* = P_{1,1}$.

Theorem 2.3.1. Define $f : P_{m,n} \to L^*$ by for all $(x, y) \in P_{m,n}$, $f((x, y)) = (x^m, y^n)$. Then f is a lattice isomorphism of $P_{m,n}$ onto L^*.

Proof. Clearly f is single valued and one-to-one. Let $(x, y) \in L^*$. Then $(x^{\frac{1}{m}}, y^{\frac{1}{n}}) \in P_{m,n}$ and so f maps $P_{m,n}$ onto L^*. Let $(x, y), (u, v) \in P_{m,n}$. Then

$$\begin{aligned} f((x, y) \wedge (u, v)) &= f((x \wedge u, y \wedge v)) \\ &= ((x \wedge u)^m, (y \wedge v)^n) \\ &= (x^m \wedge u^m, y^n \wedge v^n) \\ &= (x^m, y^n) \wedge (u^n, v^n) \\ &= f((x, y)) \wedge f((u, v)). \end{aligned}$$

A similar argument shows $f((x, y) \vee (u, v)) = f((x, y)) \vee f((u, v))$. ■

In the above table we see that for Mexico, $(0.620, 0.792) \notin P_{2,2} \cup P_{3,1} \cup P_{1,3}$. We have $(0.620, 0.792) \in P_{2,3} \cap P_{3,2}$.

For the United States, $\nexists m, n$ such that $(1^m, 0.095^n) \leq 1$.

Let $a, b, c, d \in [0, 1]$. Then $(a, b) \wedge (c, d) = (a \wedge c, b \wedge d) = (a, d) \wedge (c, b)$ even though $d \neq b$ is possible. That is, $\wedge$ is not one-to-one.

Consider Mexico again. Let P denote the set of all Pythagorean fuzzy sets. Define $g : P_{2,3} \to P$ by for all $(x, y) \in P_{2,3}$, $g((x, y)) = (x^{\frac{3}{2}}, y)$. Note $(x^{\frac{3}{2}}, y) \in P$ since $(x^{\frac{3}{2}})^2 + y^2 = x^3 + y^2 \leq 1$.

Let i be a positive real number and $x \in [0, 1]$. Suppose $x^{\frac{1}{i}} > 1$. Then $x > 1^i = 1$, a contradiction.

Let μ, ν be fuzzy subsets of a set X.

(1) $\pi(x) = 1 - \mu^2(x) - \nu^2(x)$

(2) $\pi(x) = \sqrt[2]{1 - \mu^2(x) - \nu^2(x)}$

μ, ν Pythagorean:

(1) is the uncertainty for μ^2, ν^2 since μ^2, ν^2 is intuitionistic.

(2) is the uncertainty for μ, ν since μ, ν are not intuitionistic.

Let $\mathcal{I}([0, 1]) = \{[x_1, x_2] | 0 \leq x_1 \leq x_2 \leq 1\}$. Define $\wedge, \vee$ on $\mathcal{I}([0, 1])$ by $[x_1, x_2]$, $[y_1, y_2] \epsilon \mathcal{I}([0, 1])$,

$$\begin{aligned} [x_1, x_2] \wedge [y_1, y_2] &= [x_1 \wedge y_1, x_2 \wedge y_2], \\ [x_1, x_2] \vee [y_1, y_2] &= [x_1 \vee y_1, x_2 \vee y_2]. \end{aligned}$$

Define $\leq$ on $\mathcal{I}([0, 1])$ by for all $[x_1, x_2]$, $[y_1, y_2] \epsilon \mathcal{I}([0, 1])$, $[x_1, x_2] \leq [y_1, y_2]$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

Theorem 2.3.2. Define $f : L^* \to \mathcal{I}([0, 1])$ by for all $(x_1, x_2) \epsilon L^*$, $f((x_1, x_2)) = [x_1, 1 - x_2]$. Then f is a lattice isomorphism of L^* onto $\mathcal{I}([0, 1])$.

Proof. Let $(x_1, x_2) \epsilon L^*$. Then $[x_1, 1 - x_2] \epsilon \mathcal{I}([0, 1])$ since $x_1 + x_2 \leq 1$ and so $x_1 \leq 1 - x_2$. Clearly, f is single-valued and one-to-one. Let $[x_1, x_2] \epsilon \mathcal{I}([0, 1])$. Then $x_1 + 1 - x_2 \leq 1$ and $f((x_1, 1 - x_2)) = [x_1, x_2]$. Thus f maps L^* onto $\mathcal{I}([0, 1])$. Let (x_1, x_2),

$(y_1, y_2) \epsilon L^*$. Then

$$\begin{aligned} f((x_1, x_2) \wedge (y_1, y_2)) &= f((x_1 \wedge y_1, x_2 \vee y_2)) \\ &= [x_1 \wedge y_1, 1 - x_2 \vee y_2] \\ &= [x_1 \wedge y_1, (1 - x_2) \wedge (1 - y_2)] \\ &= [x_1, 1 - x_2] \wedge [y_1, 1 - y_2] \\ &= f((x_1, x_2)) \wedge f((y_1, y_2)). \end{aligned}$$

Also

$$\begin{aligned} f((x_1, x_2) \vee (y_1, y_2)) &= f((x_1 \vee y_1, x_2 \wedge y_2)) \\ &= [x_1 \vee y_1, 1 - x_2 \wedge y_2] \\ &= [x_1 \vee y_1, (1 - x_2) \vee (1 - y_2)] \\ &= [x_1, 1 - x_2] \vee [y_1, 1 - y_2] \\ &= f((x_1, x_2)) \vee f((y_1, y_2)). \end{aligned}$$ ■

2.4 Exercises

1. Define $f : [0, 1] \to [0, 1]$ by for all $x \epsilon [0, 1]$, $f(x) = x^c$. Prove that f is a lattice isomorphism of $([0, 1], \wedge, \vee)$ onto $([0, 1], \vee, \wedge)$.

2. Let $L_2([0, 1]) = \{(x_1, x_2) \subseteq [0, 1] | 0 \leq x_1 \leq x_2 \leq 1\}$. Prove that $L_2([0, 1])$ is lattice isomorphic to L^*.

CHAPTER

Global cybersecurity[*] 3

3.1 Global cybersecurity and cybersecurity threat

The Global Society Security Index (GCI) was first launched in 2015 by the International Telecommunication Union (ITU) to measure the commitment of 193 ITU Member States and the state of Palestine to cybersecurity help them identify areas of improvement and encourage countries to take action, through raising the awareness on the state of cybersecurity worldwide. We consider the Americas. The ranking for column 1 in the following table was taken from [38, p. 28]. The SDG rankings in columns 2 and 3 were taken from [72]. The ranking in column 4 was taken from [86]. NordVPN's research partner was Statista, world's leading business data provider. Statista collected the data used in analysis and approved the methodology used to create the Cyber Risk Index.

Americas

Let A denote the Cybersecurity ranking in column 1 and B denote the SDG ranking in column 2 (see Table 3.1). Here we have $n = 27$. Let $x_1, \ldots, x_{27}$ denote the 27 countries involved. Then $M(\mu_A, \mu_B) = \sum_{i=1}^{27} \mu_A(x_i) \wedge \mu_B(x_i)/[27(28) - \sum_{i=1}^{27} \mu_A(x_i) \wedge \mu_B(x_i)]$. We find $M(\mu_A, \mu_B) = \frac{296}{460} = 0.643$. The smallest M can be $\frac{n+1}{3n-1} = \frac{28}{80} = 0.35$ since n is odd. Hence $\frac{0.643-0.350}{1-0.350} = 0.451$.

Recall $S = \frac{2M}{1+M} = \frac{2(0.643)}{1.643} = 0.783$. The smallest S can be is $\frac{2(0.35)}{1.35} = 0.519$. Thus $\frac{0.783-0.519}{1-0.519} = \frac{0.264}{0.481} = 0.549$. Consequently, in both cases the fuzzy similarity measure is medium.

We next consider the SDG $8, 9, 10$ ranking in column three. Let C denote the ranking and μ_C be the associated fuzzy subset. As before, let A denote the ranking in column one. We rerank the countries that are appear in both the A and C rankings. Then $n = 21$ and so $n(n+1) = 462$. Let $x_1, \ldots, x_{21}$ denote the countries involved. Then $M(\mu_A, \mu_C) = \frac{181}{462-182} = \frac{181}{281} = 0.644$. Since n is odd, the smallest M can be is $\frac{n+1}{3n-1} = \frac{22}{62} = 0.355$. Hence $\frac{0.644-0.355}{1-0.355} = \frac{0.289}{0.645} = 0.448$.

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00011-8

Table 3.1 Cybersecurity and risk.

Country	Global Cyberse-curity	SDG Overall	SDG 8,9,10	Cybersecu-rity Threat
USA	1	17	0.694/2	16
Canada	2	3	0.796/1	23
Brazil	3	12	0.476/12.5	50
Mexico	4	18	0.402/19	64
Uruguay	5	7	0.567/3	
Dominican Rep.	6	6	0.476/12.5	48
Chile	7	2	0.512/8	
Costa Rica	8	1	0.515/5.5	60
Columbia	9	8	0.417/18	69
Cuba	10	16		
Paraguay	11	14	0.503/9	54
Peru	12	5	0.515/5.5	70
Argentina	13	11	0.513/7	
Panama	14	19	0.445/16	78
Jamaica	15	20	0.538/4	
Suriname	16	10		
Guyana	17	26		
Venezuela	18	23	0.441/17	85
Ecuador	19	4	0.474/14	
Trinidad and Tobago	20	24		
Barbados	21			
Bolivia	22	9		
Antigua and Barbuda	23			
Bahamas	24			
El Salvador	25	21	0.447/15	
Guatemala	26	25	0.374/21	
St. Kitts and Nevis	27			
St. Vincent and the Grenadines	28			
St. Lucia	29			
Belize	30	15		
Grenada	31			
Nicaragua	32	13	0.479/10.5	84
Haiti	33	27	0.479/10.5	
Dominica	34			
Honduras	35	22	0.398/20	89

Now $S(\mu_A, \mu_C) = \frac{2(0.644)}{1.644} = 0.783$. The smallest S can be is $\frac{2(0.355)}{1.355} = 0.524$. Thus $\frac{0.783-0.524}{1-0.524} = \frac{0.259}{0.476} = 0.544$. In both cases the fuzzy similarity measure is medium.

3.2 Risk

The following ranking is from [51]. The countries are ranked from highest risk to lowest risk (see Table 3.2).

Table 3.2 Risk.

Country		Country	
Belgium	1	Qatar	30
Finland	2	Israel	31
Spain	3	Serbia	32
Denmark	4	Cyprus	33
Germany	5	Mauritius	34
Lithuania	6	Romania	35
France	7	New Zealand	36
Sweden	8	Russia	37
United Kingdom	9	North Macedonia	38
Portugal	10	Thailand	39
Netherlands	11	Slovenia	40
Poland	12	Georgia	41
Luxembourg	13	Turkey	42
Norway	14	United Arab Emirates	43
Australia	15	Iceland	44
USA	16	Uruguay	45
Croatia	17	Egypt	46
Greece	18	India	47
Slovakia	19	Dominican Republic	48
Italy	20	China	49
Malaysia	21	Brazil	50
Japan	22	Bulgaria	51
Canada	23	Ukraine	52
Singapore	24	Kazakhstan	53
Switzerland	25	Paraguay	54
Latvia	26	Philippines	55
Ireland	27	Tunisia	56
Czech Rep.	28	Nigeria	57
Hungary	29	Morocco	58

continued on next page

Table 3.2 (*continued*)

Country		Country	
South Africa	59	Cambodia	88
Costa Rica	60	Honduras	89
Bangladesh	61	Libya	90
Indonesia	62	Namibia	91
Moldova (Rep. of)	63	Myanmar	92
Mexico	64	Afghanistan	93
Saudi Arabia	65		
Albania	66		
Kenya	67		
Uganda	68		
Columbia	69		
Peru	70		
Zambia	71		
Belarus	72		
Tanzania (Rep. of)	73		
Jordan	74		
Uzbekistan	75		
Pakistan	76		
Sri Lanka	77		
Panama	78		
Armenia	79		
Kyrgyzstan	80		
Cameroon	81		
Nepal	82		
Zimbabwe	83		
Nicaragua	84		
Venezuela	85		
Ethiopia	86		
Mongolia	87		

In [86], a cyber risk index is provided for 50 countries. The risk of a country to cybercrime is determined by 14 factors (see Table 3.3).

Factors #1-13 indicated a higher risk, while factor #14 lowered the risk in a given country. It was determine in [86] that residents of developed countries are more likely to become victims of cybercrime.

In [51], the countries most at risk of cybercrime were determined. To make this determination, data from three major cybersecurity authorities was combined. These authorities were the National Cyber Security Index (NCSI), the Global Cybersecurity Index (GCI) (2020), and the Cybersecurity Exposure Index (CEI) (2020). The results were determined by finding the cybersecurity scores, all three of which were

Table 3.3 Risk to cybercrime.

	Factor		Factor
1	Urban population	8	Online games penetration
2	Monthly average wage	9	VoD penetration
3	Tourism	10	Public Wi-Fi availability
4	Internet penetration	11	Facebook penetration
5	Smartphone penetration	12	Instagram penetration
6	Time spent on internet	13	Crime index
7	E-commerce	14	Global Cybersecurity Index

expressed as percentages and assigned each of those scores to the 93 countries. The mean average of the NCSI, GCI, and CEI's total scores is referred in [51] as the Cyber-Safety Score.

We consider ranking of the 50 countries in [86]. We find the fuzzy similarity measure of the ranking in [86] and the ranking in [51]. To accomplish this, we delete the countries which do not appear in both countries and rerank them. The deletion leaves 38 countries. The Cyber Risk ranking is from high risk to low risk while the Cyber Safety ranking is from low risk for cybercrime to high. Consequently, we reverse rank the Cyber Safety rank. Let A denote the Cyber Risk ranking and B the Cyber Safety ranking (see Table 3.4).

Table 3.4 Risk and safety.

Country	Cyber Risk	Cyber-Safety
Iceland	1/1	44/31
Sweden	2/2	8/7
United Arab Emirates	3/3	43/30
Norway	4/4	14/12
United States	5/5	16/14
Singapore	6/6	24/20
Ireland	7/7	27/23
New Zealand	8/8	36/26
Denmark	9/9	4/3
United Kingdom	10/10	9/8
Israel	11/11	31/25
Finland	12/12	2/1
Belgium	14	
Canada	14/13	23/19
Chile	14	
Australia	16/14	15/13
Netherlands	17/15	11/10

continued on next page

Table 3.4 (*continued*)

Country	Cyber Risk	Cyber-Safety
Argentina	18	
Switzerland	19/16	25/21
South Korea	20	
Germany	21/17	5/4
Brazil	22/18	50/33
Austria	23	
Italy	24/19	20/16
Saudi Arabia	25.5	
Spain	25.5/20	3/2
Greece	27/21	18/15
Malaysia	28/22	21/17
Czech Rep.	29/23	28/24
France	30/24	7/6
Estonia	31	
Portugal	32/25	10/9
Mexico	33/26	64/38
Lithuania	34/27	6/5
Japan	35/28	22/18
Hungary	36	
Latvia	37/29	26/22
Turkey	38/30	42/29
Poland	39/31	12/11
Russia	40/32	37/27
Ukraine	41/33	52/34
Iran	42	
Philippines	43/34	55/35
Thailand	44/35	39/28
China	45/36	49/32
South Africa	46/37	59/37
Indonesia	47	
Iraq	48	
Nigeria	49/38	57/36
India	50	

Let X denote the set of countries. Now $S(\mu_A, \mu_B) = 1 - \sum_{x \in X} |\mu_A(x) - \mu_B(x)|/38(39) = 1 - \frac{366}{1482} = 0.753$. Since 38 is even, the smallest value S can be is $\frac{n/2+1}{n+1} = \frac{20}{39} = 0.513$. Thus $\frac{0.753-0.513}{1-0.513} = 0.493$. Now $M = \frac{S}{2-S}$. Thus $M(\mu_A, \mu_B) = \frac{0.753}{1.247} = 0.604$. The smallest M can be is $\frac{0.513}{2-0.513} = 0.345$. Thus $\frac{0.604-0.345}{1-0.345} = 0.395$. With respect to M and S, the fuzzy similarity measure is low. See Exercise 1.

3.3 Global cybercrime risk rankings

The remainder of the chapter is taken from [19]. In this section, we develop a mathematical model to examine cybercrime. We start by examining 10 countries that have been rated most prevalent in cybercrime by the Symantec Corporation, as determined by annual Internet security threat from 2006 p. 2011, [46]. Our model will depend on three major factors:

F_1: Damage

Identified as any monetary loss incurred by the use of stolen information or the costs associated with correcting such effects. All figures will be in the United States dollars

F_2: Prevalence

The relative global frequency of certain computer attacks and the frequency of online adults who suffer negative effects from a cybercrime attack.

F_3: The growth characteristics of certain types of cyber attacks from 2006 to 2011.

We next present subfactors that will be used by both F_2 and F_3.

f_1: Bot-infected computer attacks

Bots are programs installed on a compromised machine to allow an attacker to remotely control it via a communication channel and orchestrate other attacks.

f_2: Hacking

Hacking is any attack aimed at gaining access to computer systems or networks for the purpose of data mining and system manipulation. Those who use this method are greatly aided by currently available hacker software such as L0phtCrack, which is used for password checking.

f_3: Malicious code

Malicious code samples, or vandals, are auto-executable applications that are used to attack network drives. Their behavior is dependent on the code itself, but common methods are the lifting of data and passwords or gaining access to email for the use of spamming.

f_4: Phishing website hosting

Phishing attacks occur when an attacker attempts to gain confidential information (credit card number, banking information, etc.) from victims by mimicking a specific company or brand.

f_5: Spam zombies

Spanning is the delivery of unsolicited email, which may contain Trojans, viruses, and phishing attempts. Spam zombies, similar to bio-infected computers, are machines remotely controlled by an attacker, who uses it to distribute spam without the victim's knowledge.

F_3 adds another factor to be used as a multiplicand.

f_6: Adult victims

The prevalence of adult victims is identified as the percentage of online adults in each country who have suffered some detrimental effect at the result of a cybercrime in 2011.

Subfactors for F_1 are

f_1: Direct cash
f_2: Time lost cost

Expert opinion

The rating to follow for the above factors were determined faculty and staff with expertise in security technology. Their responses indicated the relative risk of cybercrime for each country in set S.

We use three different methods to achieve our goal, these methods are the Analytic Hierarchy Process (AHP), Guiasu, and Yen. Using the expert opinion, we have the following table:

$W =$

	E_1	E_2	E_3	E_4	Row Avg
F_1	0.3	0.9	0.5	0.3	0.5
F_2	0.5	0.7	0.8	0.6	0.65
F_3	0.5	0.8	0.7	0.8	0.7

Using the explanation in Chapter 1, we obtain the AHP equation

$$G = 0.27F_1 + 0.35F_2 + 0.38F_3$$

the Guiasu table and equation

	E_1	E_2	E_3	E_4	Row Avg
F_1	0.23	0.375	0.25	0.18	0.259
F_2	0.38	0.29	0.40	0.35	0.355
F_3	0.38	0.33	0.35	0.47	0.382

$$G' = 0.26F_1 + 0.35F_2 + 0.38F_3$$

and the Yen table and equation

	E_1	E_2	E_3	E_4	Row Avg
F_1	0.61	1	0.625	0.38	0.654
F_2	1	0.77	1	0.74	0.877
F_3	1	0.88	0.86	1	0.935

$$G'' = 0.27F_1 + 0.35F_2 + 0.39F_3.$$

See Exercise 2.

Subfactors

The data from each sub-factor of F_1 is in terms of monetary U. S. dollars. Since dollars are inherently equal to each other, we do not assign differing weights to f_1 and f_2. Thus, we obtain

AHP: $F_1 = 0.5f_1 + 0.5f_2$
Guiasu: $F_1 = 0.5f_1 + 0.5f_2$
Yen: $F_1 = 0.5f_1 + 0.5f_2$

We next consider F_2. The equation for F_2 will include both a weighted and constant component. The weighted component is determined by expert opinion. This will be multiplied by f_6, which is a constant, nonarbitrary factor which describes the prevalence of online victims in each country

	E_1	E_2	E_3	E_4	Row Avg
f_1	0.3	0.8	0.7	0.3	0.525
f_2	0.5	0.5	0.7	0.4	0.525
f_3	0.4	0.6	0.6	0.2	0.45
f_4	0.2	0.8	0.4	0.6	0.5
f_5	0.1	0.3	0.4	0.7	0.375

AHP: $F_2 = (0.2211f_1 + 0.2211f_2 + 0.185f_3 + 0.2105f_4 + 0.1579f_5)f_6$
Guiasu: $F_2 = (0.2233f_1 + 0.2329f_2 + 0.1930f_3 + 0.2039f_4 + 0.1569f_5)f_6$
Yen: $F_2 = (0.2280f_1 + 0.2301f_2 + 0.1938f_3 + 0.2036f_4 + 0.1545f_5)f_6$

We next consider F_3.

	E_1	E_2	E_3	E_4	Row Avg
f_1	0.2	0.7	0.9	0.3	0.525
f_2	0.5	0.5	1	0.6	0.65
f_3	0.2	0.6	0.7	0.7	0.55
f_4	0.2	0.6	0.5	0.3	0.4
f_5	0.2	0.3	0.1	0.3	0.225

AHP: $F_3 = 0.2234f_1 + 0.2766f_2 + 0.2340f_3 + 0.1702f_4 + 0.0957f_5$
Guiasu: $F_3 = 0.2077f_1 + 0.2887f_2 + 0.2283f_3 + 0.1671f_4 + 0.1082f_5$
Yen: $F_3 = 0.2132f_1 + 0.2790f_2 + 0.2310f_3 + 0.1708f_4 + 0.1060f_5$

To determine F_3 data, linear regression on the rankings of each country in set S was used. Those countries with the most negative slope—meaning they are ascending from a low rank to a rank, i.e., 10th to 1st over the course of six years—were granted the highest rankings and those with most positive were given the lowest rankings. These rankings can be found in [19].

Degree of expert opinion

We use fuzzy preference relations to examine the degree to which each major factor is preferred over other factors. We use matrix W.

Let $R_k = [\rho_{ij}^k]$, where $\rho_{ij}^k(F_i, F_j) = (e_{ik} - e_{jk} + 0.5) \wedge 1$ if $e_{ik} \geq e_{jk}$ and $\rho_{ij}^k(F_i, F_j) = 1 - [(e_{jk} - e_{ik} + 0.5) \wedge 1]$ if $e_{ik} < e_{jk}$, $i, j = 1, 2, 3$ and $k = 1, ..., 3, 4$. Then

$R_1 =$

	F_1	F_2	F_3
F_1	0.5	0.3	0.3
F_2	0.7	0.5	0.5
F_3	0.7	0.5	0.5

and $R_2 =$

	F_1	F_2	F_3
F_1	0.5	0.7	0.6
F_2	0.3	0.5	0.4
F_3	0.4	0.6	0.5

$R_3 =$

	F_1	F_2	F_3
F_1	0.5	0.2	0.3
F_2	0.8	0.5	0.6
F_3	0.7	0.4	0.5

and $R_4 =$

	F_1	F_2	F_3
F_1	0.5	0.2	0
F_2	0.8	0.5	0.3
F_3	1	0.7	0.5

Let $A_k = [a_{ij}^k]$, where $a_{ij}^k = 1$ if $\rho_{ij}^k > 0.5$ and $i, j = 1, 2, 3$ and $k = 1, 2, 3, 4$.

Let $R = [r_{ij}]$, where for all $i, j = 1, 2, 3$, $r_{ij} = \frac{1}{4} \sum_{k=1}^{4} a_{ij}^k$ if $i \neq j$ and 0 otherwise. Then

$R =$

	F_1	$\dot{F}_2$	F_3
F_1	0	1/4	1/4
F_2	3/4	0	1/4
F_3	3.4	2/4	0

Let $G = [g_{ij}]$, $g_{ij} = 1$ if $r_{ij} > 0.5$ and $r_{ij} > 0.5$ and 0 otherwise. Then g_{ij} expresses whether F_i defeats F_j.

Now $g_i = \frac{1}{2} \sum_{j=1}^{3} g_{ij}$ is the mean degree to which F_i is preferred to all other F_j. We find that $g_1 = 0$, $g_2 = \frac{1}{2}$, and $g_3 = \frac{1}{2}$.

Let z_Q^i be the fuzzy consensus winner, or the extent to which F_i is preferred to Q other $F_j, i, j = 1, 2, 3$ and Q denotes most. We define $z_Q^i = \mu_Q(g_i) = 1$ if $0.8 \leq g_i \leq 1, 2g_i - 0.6$ if $0.3 < g_i < 0.8$, and 0 if $0 \leq g_i \leq 0.3$. We find that

$$z_Q^1 = \mu_Q(g_1) = \mu_Q(0) = 0,$$
$$z_Q^2 = \mu_Q(g_2) = \mu_Q(\frac{1}{2}) = 2(\frac{1}{2}) = 0.4,$$
$$z_Q^3 = \mu_Q(g_3) = \mu_Q(\frac{1}{2}) = 2(\frac{1}{2}) = 0.4,$$

Fuzzy preference relations were used further to determine relationships between each of the factors.

Let $S, I, >, R, \sim$ be relations on X having the following meanings:
S: outranking relation
xSy means x is not worse than y
I: indifference relation
xIy means x and y are indifferent
R: incompatibility relation
xRy means x and y are incomparable
$\succ$: preference relation
$x \succ y$ means x is preferred to y
$\sim$: nonpreference relation
$x \sim y$ means that x and y cannot be discriminated against
$I(S)(x, y) = S(x, y) \wedge S(y, x)$
$R(S)(x, y) = (1 - S(x, y)) \wedge (1 - S(y, x))$
$\succ (S)(x, y) = S(x, y) \wedge (1 - S(y, x))$
$\sim (S)(x, y) = (S(x, y) \wedge S(y, x)) \vee ((1 - S(x, y)) \wedge (1 - S(y, x)))$
Let $S = R$, where

$R =$

	F_1	F_2	F_3
F_1	0	$\frac{1}{4}$	$\frac{1}{4}$
F_2	$\frac{3}{4}$	0	$\frac{1}{4}$
F_3	$\frac{3}{4}$	$\frac{2}{4}$	0

Then

$I(S) =$

	F_1	F_2	F_3
F_1	0	$\frac{1}{4}$	$\frac{1}{4}$
F_2	$\frac{1}{4}$	0	$\frac{1}{4}$
F_3	$\frac{1}{4}$	$\frac{1}{4}$	0

, $R(S) =$

	F_1	F_2	F_3
F_1	1	$\frac{1}{4}$	$\frac{1}{4}$
F_2	$\frac{1}{4}$	1	$\frac{2}{4}$
F_3	$\frac{1}{4}$	$\frac{2}{4}$	1

and

$\succ (S) =$

	F_1	F_2	F_3
F_1	0	$\frac{1}{4}$	$\frac{1}{4}$
F_2	$\frac{3}{4}$	0	$\frac{1}{4}$
F_3	$\frac{3}{4}$	$\frac{2}{4}$	0

, $\sim (S) =$

	F_1	F_2	F_3
F_1	1	$\frac{1}{4}$	$\frac{1}{4}$
F_2	$\frac{1}{4}$	1	$\frac{2}{4}$
F_3	$\frac{1}{4}$	$\frac{2}{4}$	1

For $m, n = 1, 2, 3, 4$ define $\nu(n, m) = 1$ if $|\rho_{ij}^{m} - \rho_{ij}^{n}| \leq 1 - 0.94 = 0.06$ and 0 otherwise. Then

$\nu(1, 2) =$

	F_1	F_2	F_3
F_1	1	0	0
F_2	0	1	0
F_3	0	0	1

$\nu(1, 3) =$

	F_1	F_2	F_3
F_1	1	0	1
F_2	0	1	0
F_3	101		

$\nu(1, 4) =$

	F_1	F_2	F_3
F_1	1	0	0
F_2	0	1	0
F_3	0	0	1

$\nu(2,3) =$

	F_1	F_2	F_3
F_1	1	0	0
F_2	0	1	0
F_3	0	0	1

$\nu(2,4) =$

	F_1	F_2	F_3
F_1	1	0	0
F_2	0	1	0
F_3	0	0	1

$\nu(3,4) =$

	F_1	F_2	F_3
F_1	1	1	0
F_2	1	1	0
F_3	0	0	1

The degree of agreement of all pairs of experts m, n to their preference is by $\nu_B(m,n) = \frac{1}{3}\sum_{i=1}^{2}\sum_{j=1+1}^{3}\nu_{ij}(m,n)$. Thus we have

$$\nu_B(1,2) = 0,\ \nu_B(1,3) = \frac{1}{3},\ \nu_B(1,4) = 0$$
$$\nu_B(2,3) = 0,\ \nu_B(2,4) = 0,\ \nu_B(3,4) = 0$$

Note $\nu_B(1,3) = \frac{1}{3}(\nu_{12}(1,3) + \nu_{13}(1,3) + \nu_{23}(1,3)) = \frac{1}{3}(0+1+0)$.

The degree of agreement of all pairs of experts m, n is given by the formula

$$\nu_B = \frac{1}{6}\sum_{m=1}^{3}\sum_{n=m+1}^{4}\nu_B(m,n).$$

We find that $\nu_B = \frac{1}{6}(0 + \frac{1}{3} + 0 + 0 + 0 + \frac{1}{3}) = \frac{1}{9}$.

3.4 Exercises

1. Investigate why the fuzzy similarity measure in Section 3.2 (Table 3.4) is low.

2. Put in the details for determining the AHP, Guiasu, and Yen tables and equations in Section 3.3.

CHAPTER

Terrorism and bioterrorism✪ 4

4.1 Global terrorism index

The Global Terrorism Index (GTI) analyses the impact of terrorism for 163 countries, which covers 99.7% of the world's population. The GTI defines terrorism as the systematic threat or use of violence by non-state actors, whether for or in opposition to established authority, with the intention of communicating a political, religious, or ideological message to a group larger than the group by generating fear and so altering (or attempting to alter) the behavior of the larger.

Table 4.1 presents the ranking of countries by GTI with respect to the impact of terrorism, [42].

The remaining countries were tied. They had no impact or were not included.

4.2 Terrorism: Interpol global policing goals and SDGs

The following depends heavily on [74]. We determine how well the Organization for Economic Cooperation and Development (OECD) country is achieving the Interpol goals with respect to the SDGs pertinent to them. We found that the Scandinavian countries were at the top in the achievement. In combating terrorism, in general, we found that the Scandinavian countries were at the top. We also show that the fuzzy similarity measure of the ranking of countries used by the SDGs relevant to terrorism compared with SDG 16 alone was high. Let A, B, and C be rankings of a set X and μ_A, μ_B, and μ_C the associated fuzzy subsets, respectively. Consider the similarity measures S and M, [75]. Suppose that $S(\mu_A, \mu_B)$ and $S(\mu_A, \mu_C)$ are known, but $S(\mu_B, \mu_C)$ is unknown. It is shown in [71] that if $S(\mu_A, \mu_B)$ is near 1, then $|S(\mu_A, \mu_C) - S(\mu_B, \mu_C)|$ is near 0. In this chapter, a similar result for M is shown.

In [47], seven global policing goals are presented that reflect Interpol's priorities against criminal and terrorist threats, in alignment with the United Nations 2030 Agenda for Sustainable Development. Interpol developed seven Global Policing Goals (GPGs) to address a range of issues related to crime and security. Endorsed

✪ This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00012-X

Table 4.1 Impact of terrorism.

Country	Rank	Country	Rank	Country	Rank
Afghanistan	1	USA	30	Belgium	59
Burkino Faso	2	Greece	31	Spain	60
Somalia	3	Libya	32	Austria	61
Mali	4	Palestine	33	Japan	62
Syria	5	France	34	Saudi Arabia	63
Pakistan	6	Germany	35	Sweden	64
Iraq	7	Nepal	36	Switzerland	65
Nigeria	8	Algeria	37	Ecuador	66
Myanmar	9	Tanzania	38	Netherlands	67
Niger	10	Burundi	39	Jordan	68
Cameroon	11	Tunisia	40	Australia	69
Mozambique	12	Peru	41	Uzbekistan	70
India	13	UK	42	Paraguay	71
Dem. Rep. Congo	14	Bangladesh	43	Mexico	72
Columbia	15	Djibouti	44	Ukraine	73
Egypt	16	Russia	45	Cyprus	74
Chile	17	New Zealand	46	Malaysia	75
Philippines	18	Cote d'Ivoire	47	UAE	76
Chad	19	Uganda	48	Senegal	77
Kenya	20	Norway	49	Eswatini	78
Iran	21	Tajikistan	50	Bahrain	80.5
Yemen	22	Venezuela	**51**	Rwanda	80.5
Turkey	23	Lebanon	52	South Africa	80.5
Indonesia	24	Italy	53	Uruguay	80.5
Israel	25	Canada	54	Morocco	83
Thailand	26	Central African Rep.	55	Romania	84
Togo	27	Ethiopia	56	Brazil	85
Benin	28	Argentina	57	Lithuania	86
Sri Lanka	29	Slovakia	58	Ireland	87.5
Mauritania	87.5	Vietnam	89	Angola	90.5
Denmark	90.5	Kosovo	90.5		

by their member countries in 2017, the Goals were officially launched in 2018. The description of the Goals can be found in the Appendix. The SDGs related to these goals appear in Table 4.2.

We determine how well an OECD country is achieving the Interpol goals with respect to the SDGs pertinent to them. The OECD is an international organization of 38 countries committed to democracy and the market economy. The OECD's mission is to bring together the governments committed to democracy and the market economy from around the world.

Table 4.2 Interpol goal and related SDGs.

Interpol Goal	SDGs
I_1	9,11,16,17
I_2	5,8,9,10,16,17
I_3	5,8,10,16
I_4	9,16,17
I_5	11,16,17
I_6	2,3,8,11,12,15,16
I_7	1,2,3,6,8,11,12,13,14,15,16

4.3 Achievement of goals

Tables 4.3 and 4.4 below provide values for how well an OECD country is achieving the Interpol Goals. A country with the superscript * is one in which the Interpol Goal I_7 was rated na and a country with a superscript # was one with na for I_2 and I_3, see Table 4.3. How the values in Table 4.3 below are determined is illustrated in the following example. Consider Australia and Goal I_1. The SDGs involved for I_1 are 9, 11, 16, and 17. We obtain $77.9 = \frac{1}{4}(84.2 + 80.6 + 85.7 + 61.1)$, where 84.2, 80.6, 85.7, 61.1 are the values from [109] denoting how well Australia is achieving SDG 9, SDG 11, SDG16, SDG 17, respectively.

4.4 Fuzzy similarity measures

For ease of reading, we recall some results previously stated.

We apply fuzzy similarity measures to rankings of members of a finite set. Suppose that X is a finite set with n elements. Let A be a one-to-one function of X onto $\{1, 2, ..., n\}$. Then A is called a ranking of X. Define the fuzzy subset μ_A of X as follows: $\forall x \in X$, $\mu(x) = A(x)/n$. We wish to consider the similarity of two rankings of X by the use of fuzzy similarity measures.

Let μ_A, μ_B be the fuzzy subsets of X associated with two rankings A and B of X, respectively. Then M and S are fuzzy similarity measures, where

$$M(\mu_A, \mu_B) = \frac{\sum_{x\in X} \mu_A(x) \wedge \mu_B(x)}{\sum_{x\in X} \mu_A(x) \vee \mu_B(x)},$$

$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x\in X} |\mu_A(x) - \mu_B(x)|}{\sum_{x\in X} \mu_A(x) + \mu_B(x)}.$$

Theorem 4.4.1. *[71,75]* (1) *Suppose that n is even. Then the smallest value M can be is*

$$M = \frac{n+2}{3n+2}.$$

Table 4.3 Interpol policing goal scores/rank.

Country	I_1	I_2	I_3	I_4	I_5	I_6	I_7
Australia	77.90/12	75.62/15	80.68 /17	77.00/10	75.80/21	69.29/34	70.11/29
Austria	81.50/7	81.45/7	85.12/9	80.06 /16	81.93/7	77.60/17	na
Belgium	76.85/16	80.63/8	86.40/7	75.03/14	77.17/16	78.09/13	76.26 /20
Canada	77.08/15	78.52/12	82.82 /11	75.97/12	77.97/13	74.04/25	75.43 /22
Chile	71.30/25	63.83/32	63.60/32	68.70/22	78.67/11	74.14/24	79.58 /6
Czech Rep.	72.70/23	74.98/17	82.80 /12	67.17/26	75.87/20	82.07/1	na
Denmark	90.225/2	89.32/2	89.50/3	90.23/2	90.97/1	81.19/2	81.61 /1.5
Estonia	73.78/20	72.85/19	80.02 /19	68.27/24	77.87/14	79.89/3	81.61 /1.5
Finland	84.72/4	86.70/4	90.62/1	83.53 /4	85.07/5	78.41/11	78.89/14
France	78.08/11	79.25/10	81.70 /14	75.10/13	79.57/10	76.01/20	79.10/11
Germany	84.45/5	81.95/6	82.05/13	82.30/15	85.80/4	78.89/7	79.26 /9
Greece	64.60/34	58.80/33	62.32 /33	58.77/34	72.83/24	69.63/32	74.21/25
Hungary	65.15/32.5	66.05/30	73.80 /26	58.17/30	70.33/31	78.57/10	na
Iceland	81.45/8	83.93/5	90.18/2	78.63 /7	83.43/6	72.93/28	74.82/24
Ireland	68.88/28	72.77/20	84.00 /10	63.67/28	69.43/35	79.53/6	80.32 /3
Israel	71.52/24	69.40/26	71.00 /29	68.67/23	69.53/34	69.31/33	69.75/30
Italy	69.02/27	70.32/22.5	73.75 /27	67.37/25	70.77/30	74.56/22	75.44/21
Japan	77.62/14	76.48/14	78.52/20	78.37/8	76.87/17	77.53/18	79.11 /10
Korea, Rep.	73.20/21	74.85/18	78.00 /22	70.83/19	69.70/33	76.13/19	77.81 /18
Latvia	65.75/31	67.78/27.5	76.75 /24	58.90/33	71.23/27	78.80/8	79.82 /4
Lithuania	65.15/ 32.5	67.78/27.5	70.68 /30	59.17/32	71.73/25	77.86/15	79.61/5
Luxembourg	78.12/10	65.56/31	80.75 /16	72.67/16	81.03/8	71.37/30	na
Mexico	57.70/36	52.43/35	54.52 /35	49.87/36	64.83/36	67.19/36	72.45/27
Netherlands	77.65/13	79.78/9	85.68 /8	73.17/15	76.10/19	78.10/12	78.94 /12

continued on next page

Table 4.3 (*continued*)

Country	I_1	I_2	I_3	I_4	I_5	I_6	I_7
New Zealand	78.6/9	na	na	77.13/9	80.17/9	74.29/23	78.11/16
Norway	87.65/3	88.45/3	87.78/5	88.17/3	90.20/3	71.16/31	73.25/26
Poland	67.05/29	66.48/29	72.80/28	63.23/29	71.10/28	79.83/5	79.42/8
Portugal	70.82/26	69.87/24	76.10/25	64.53/27	75.73/22	75.30/21	77.83/17
Slovak Rep.	66.68/30	69.63/25	78.25/21	61.57/31	71.00/29	78.76/9	na
Slovenia	73.15/22	77.78/13	87.025/6	68.90/21	77.20/21	79.90/4	78.72/15
Spain	74.22/19	72.48/21	76.925/23	69.27/20	76.27/20	73.60/26	77.65/19
Sweden	91.00/1	91.02/1	89.05/4	91.23/1	90.77/1	78.01/14	78.92/13
Switzerland	81.98/6	78.60/11	81.25/15	76.53/11	78.20/11	72.44/29	na
Turkey	63.95/35	57.62/34	57.10/34	61.80/30	69.77/30	68.40/35	70.70/28
U. K.	76.70/17	75.27/16	80.32/18	72.00/18	75.13/18	76.70/16	79.44/7
U. S.	74.52/18	70.32/22.5	70.60/31	72.03/17	71.60/17	73.24/27	74.87/23

(2) *Suppose that n is odd. Then the smallest value M can be is*

$$M = \frac{n+1}{3n-1}.$$

Theorem 4.4.2. *[71,75]* (1) *Suppose that n is even. Then the smallest value S can be is*

$$S = \frac{n/2+1}{n+1}.$$

(2) *Suppose that n is odd. Then the smallest value S can be is*

$$S = \frac{1}{2} + \frac{1}{2n}.$$

The equations in the above theorems give the smallest value $M(\mu,\nu)$ and $S(\mu,\nu)$ can take on. In general, $M(\mu,\nu)$ and $S(\mu,\nu)$ are bounded below by $\frac{1}{3}$ and $\frac{1}{2}$, respectively. If we wish to calculate a value for $M(\mu,\nu)$ and $S(\mu,\nu)$ in which the values are bounded below by 0, we can use the following formulas:

$$\frac{M(\mu,\nu) - \frac{n+2}{3n+2}}{1 - \frac{n+2}{3n+2}} \quad \text{and} \quad \frac{S(\mu,\nu) - \frac{n/2+1}{n+1}}{1 - \frac{n/2+1}{n+1}}$$

if n is even and

$$\frac{M(\mu,\nu) - \frac{n+1}{3n-1}}{1 - \frac{n+1}{3n-21}} \quad \text{and} \quad \frac{S(\mu,\nu) - (\frac{1}{2} + \frac{1}{2n})}{1 - (\frac{1}{2} + \frac{1}{2n})}$$

if n is odd.

Describing fuzzy similarity values linguistically, one might say the similarity is very low if the value is between 0 and 0.2, low if the value is between 0.2 and 0.4, medium if the value is between 0.4 and 0.6, high if the value is between 0.6 and 0, 8, and very high if the value is between 0.8 and 1.

Let A, B, and C be rankings of a set X of n elements. Suppose that $S(\mu_A,\mu_B)$ and $S(\mu_A,\mu_C)$ are known, but $S(\mu_B,\mu_C)$ is unknown. The following result is given in [75].

Proposition 4.4.3. $|S(\mu_{B,}\mu_C) - S(\mu_A,\mu_C)| \le 1 - S(\mu_A,\mu_B)$.

Let $\epsilon = 1 - S(\mu_A,\mu_B)$. We wish to determine $|M(\mu_{B,}\mu_C) - M(\mu_A,\mu_C)|$ if $M(\mu_A,\mu_B)$ and $M(\mu_A,\mu_C)$ are known, but $M(\mu_B,\mu_C)$ is unknown. We know by [109] that $M = \frac{S}{2-S}$.

Proposition 4.4.4. $|M(\mu_{B,}\mu_C) - M(\mu_A,\mu_C)| < \epsilon'$, *where* $\frac{8}{9}\epsilon \le \epsilon' \le 2\epsilon$.

Proof. We have that

$$\begin{aligned}
&|M(\mu_B,\mu_C)-M(\mu_A,\mu_C)| \\
&=|\frac{S(\mu_B,\mu_C)}{2-S(\mu_B,\mu_C)}-\frac{S(\mu_A,\mu_C)}{2-S(\mu_A,\mu_C)}| \\
&=|\frac{S(\mu_B,\mu_C)(2-S(\mu_A,\mu_C))-S(\mu_A,\mu_C)(2-S(\mu_B,\mu_C))}{(2-S(\mu_B,\mu_C))((2-S(\mu_A,\mu_C)))}| \\
&=\frac{2|S(\mu_B,\mu_C)-S(\mu_A,\mu_C)|}{(2-S(\mu_B,\mu_C))((2-S(\mu_A,\mu_C)))} \\
&<\frac{2}{(2-S(\mu_B,\mu_C))((2-S(\mu_A,\mu_C))}\epsilon,
\end{aligned}$$

where $\varepsilon = 1 - S(\mu_A, \mu_B)$. By [4], the smallest S can be is $\frac{1}{2}$ (in the limit). Thus the smallest $\frac{2}{(2-S(\mu_B,\mu_C))((2-S(\mu_A,\mu_C))}$ can be $\frac{8}{9}$. The largest S can be is 1. Hence, the largest $\frac{2}{(2-S(\mu_B,\mu_C))((2-S(\mu_A,\mu_C))}$ is 2. ■

We next consider rankings on a set with n elements

Suppose n is even. Then the smallest S can be is $\frac{\frac{n}{2}+1}{n+1}$ by [71]. Hence,

$$|\frac{S-\frac{\frac{n}{2}+1}{n+1}}{1-\frac{\frac{n}{2}+1}{n+1}}-\frac{S'-\frac{\frac{n}{2}+1}{n+1}}{1-\frac{\frac{n}{2}+1}{n+1}}|=\frac{|S-S'|}{1-\frac{\frac{n}{2}+1}{n+1}}<(2+\frac{1}{n})\epsilon.$$

Suppose n is odd. Then the smallest S can be is $\frac{1}{2}+\frac{1}{2n}$ by [71]. Hence,

$$\begin{aligned}
&|\frac{S-(\frac{1}{2}+\frac{1}{2n})}{1-(\frac{1}{2}+\frac{1}{2n})}-\frac{S'-(\frac{1}{2}+\frac{1}{2n})}{1-(\frac{1}{2}+\frac{1}{2n})}| \\
&=\frac{|S-S'|}{\frac{1}{2}-\frac{1}{2n}}=\frac{2|S-S'|}{1-\frac{1}{n}}<\frac{2}{1-\frac{1}{n}}\epsilon.
\end{aligned}$$

Suppose n is even. Then the smallest M can be is $\frac{n+2}{3n+2}$ by [71]. Hence,

$$\begin{aligned}
&|\frac{M-\frac{n+2}{3n+2}}{1-\frac{n+2}{3n+2}}-\frac{M'-\frac{n+2}{3n+2}}{1-\frac{n+2}{3n+2}}| \\
&=\frac{|M-M'|}{1-\frac{n+2}{3n+2}}=\frac{|M-M'|}{\frac{2n}{3n+2}}=\frac{3+\frac{2}{n}}{2}|M-M'|.
\end{aligned}$$

Suppose n is odd. Then the smallest M can be is $\frac{n+1}{3n-1}$ by [71]. Hence,

$$|\frac{M-\frac{n+1}{3n-1}}{1-\frac{n+1}{3n-1}}-\frac{M'-\frac{n+1}{3n-1}}{1-\frac{n+1}{3n-1}}|=\frac{|M-M'|}{1-\frac{n+1}{3n-1}}=\frac{|M-M'|}{\frac{2n-2}{3n-1}}=\frac{3-\frac{1}{n}}{2-\frac{2}{n}}|M-M'|.$$

4.5 Scores and ranks by averages

In Table 4.4, we provide Interpol Goal average scores and ranks of countries. For the weighted average the weights are determined as follows: the number of times an SDG appears in Table 4.1 is 38. The weight of I_i is the number SDGs appearing for I_i divided by 38, $i = 1, \ldots, 7$. For example, the weight for I_1 is $4/38$.

Table 4.4 Interpol policing goal average scores/rank.

Country	Weighted Average/Rank	Average/Rank
Australia	73.75/23	75.20/19
Austria	80.90*/5	81.28/6
Belgium	77.76/12	78.63/8
Canada	76.86/18	77.40/15
Chile	72.60/30	71.40/29
Czech Rep.	76.87*/17	75.93/18
Denmark	85.91/1	87.58/1
Estonia	77.57/14	76.33/17
Finland	82.74/3	83.99/3
France	78.44/8	78.40/10
Germany	81.21/4	82.10/5
Greece	67.34/34	65.88/34
Hungary	69.91*/32	68.68/33
Iceland	79.21/7	80.77/7
Ireland	75.99/19	74.08/23
Israel	69.83/33	69.88/32
Italy	72.61/29	71.60/27
Japan	77.95/11	77.79/12
Korea, Rep	75.37/20	74.36/21
Latvia	73.39/26	72.03/26
Lithuania	72.67/28	70.28/31
Luxembourg	73.69*/24	74.92/20
Mexico	62.49/36	59.86/36
Netherlands	78.91/9	78.49/9
New Zealand	77.34#/15	77.66/13
Norway	80.83/6	83.81/4
Poland	73.54/25	71.42/28
Portugal	73.97/22	72.88/24
Slovak Rep.	72.09*/31	70.98/30
Slovenia	78.16/10	77.52/14

continued on next page

Table 4.4 (*continued*)

Country	Weighted Average/Rank	Average/Rank
Spain	74.88/21	74.35/22
Sweden	84.91/2	87.14/2
Switzerland	77.62*/13	78.17/11
Turkey	65.29/35	65.62/35
United Kingdom	77.15/16	76.51/16
United States	72.88/27	72.46/25

4.6 Terrorism and the SDGs

It is stated in [4] that no religious pretext can ever excuse violent methods. At the same time, we will never be able to defeat terrorism long term unless we address conditions conducive to its spread. Several Security Council resolutions pertaining to the most serious threats to international peace and security generally, and to terrorism specifically, have underlined this, inter alia, in Security Council resolutions. The first pillar of the United Nations Global Counter-Terrorism Strategy also resolves to address conditions conducive to the spread of terrorism. More recently, the security-General's Plan of Action to Prevent Extremism elaborates on what some of these conditions may be lack of socioeconomic opportunities, marginalization and discrimination, poor governance violations of human rights and the rule of law, prolonged and unresolved conflicts, and radicalization in prisons.

Crucial to the rights of the SDGs as a means to galvanize the international community's efforts to tackle serious developmental-related challenges, the SDGs can also directly and indirectly help our efforts to counter terrorism by addressing conditions conducive to its spread.

SDG 16 is the most relevant SDG with respect to terrorism. In [71], SDGs 1, 4, 5, 8, 10, and 16 are listed as the SDGs relevant to terrorism. We next determine scores of how well countries are achieving these SDGs. We pay individual attention to SDG 16 and SDG 5. SDG 16 is the strongest of all SDGs with respect to combating terrorism and SDG 5 has been singled out as the SDG most pertinent to terrorism involving women.

The scores in the SDGs Ref. 5 column of Table 4.5 are determined by taking the average of how well a country is achieving the SDGs given in [71].

We next find the fuzzy similarity measure of μ and ν, where μ represents the ranking in column 1 and ν represents the ranking in column 3. We find that $M(\mu,\nu) = \frac{572.5}{759.5} = 0.754$ and $S(\mu,\nu) = 1 - \frac{183}{1332} = 0.863$. Now $n = 36$ is even. The smallest M can be is $\frac{n+2}{3n+2} = \frac{38}{110} = 0.345$ and the smallest S can be is $\frac{\frac{n}{2}+1}{n+1} = \frac{19}{37} = 0.514$. Now $\frac{0.754-0.345}{1=0.345} = \frac{0.400}{0.655} = 0.611$ and $\frac{0.863-0.514}{1-0.514} = \frac{0.349}{0.486} = 0.718$. We see that the fuzzy similarity measures are high.

Table 4.5 SDG achievement.

Country	SDGs Ref. 5/Rank	SDG 5/Rank	SDG 16/Rank
Australia	85.75/20	78.9/16	85.7/12.5
Austria	89.38/10	79.1/15	92.0/4
Belgium	89.97/8	83.9/8	86.9/11
Canada	88.40/12	80.4/14	88.1/8.5
Chile	74.35/33	70.5/29	75.9/29
Czech Rep.	87.82/13	71.1/27.5	81.2/22
Denmark	92.65/4	84.8/6	82.7/20
Estonia	85.85/19	75.3/19.5	87.8/10
Finland	93.53/1	89.2/1	92.9/2
France	87.28/14	86.5/4	76.6/26
Germany	86.13/17.5	77.0/18	83.4/18
Greece	72.68/34	62.6/34	72.8/34
Hungary	80.75/29	64.1/32	73.4/33
Iceland	92.97/2	85.5/5	93.0/1
Ireland	88.48/11	73.1/24	90.4/6
Israel	80.00/30	75.2/21	73.6/32
Italy	81.65/27	71.2/26	75.2/31
Japan	85.20/21	58.5/35	90.3/7
Korea Rep.	84.47/22	63.9/33	75.4/30
Latvia	83.57/23	70.2/30	77.0/27
Lithuania	79.97/31	72.1/25	80.5/24
Luxembourg	86.27/16	74.6/22	90.2/5
Mexico	66.37/36	77.4/17	53.1/36
Netherlands	89.42/9	81.5/11	83.5/17
New Zealand	92.70*/3	84.7/7	92.6/3
Norway	91.75/6	87.7/3	84.9/14
Poland	80.82/28	71.1/27.5	81.4/21
Portugal	83.10/25	80.7/13	84.1/15
Slovak Rep.	82.50/26	68.9/31	79.9/25
Slovenia	90.73/7	75.3/19.5	88.1/8.5
Spain	83.53/24	82.7/9	80.6/23
Sweden	92.42/5	88.9/2	83.8/16
Switzerland	86.13/17.5	82.2/10	83.0/19
Turkey	70.27/35	45.3/36	68.1/35
United Kingdom	86.73/15	81.3/12	85.7/12.5
United States	78.43/32	73.4/23	76.1/28

4.7 Appendix: the Interpol global policing goals and the SDGs

Interpol global policing goals

Goal 1. Counter the threat of terrorism
Goal 2. Promote border integrity
Goal 3. Protect vulnerable communities
Goal 4. Secure cyberspace for people and businesses
Goal 5. Promote global integrity
Goal 6. Curb illicit markets
Goal 7. Support environmental security and sustainability

Sustainable development goals

Goal 1. End poverty in all its forms everywhere

Goal 2. End hunger, achieve food security, improved nutrition, and promote sustainable agriculture

Goal 3. Ensure healthy lives and promote well-being for all at all ages

Goal 4. Ensure inclusive and equitable quality education and promote lifelong learning opportunities for all

Goal 5. Achieve gender equality and empower all women and girls

Goal 6. Ensure availability and sustainable management of water and sanitation for all

Goal 7. Ensure access to affordable, reliable, sustainable and modern energy for all

Goal 8. Promote sustained, inclusive and sustainable growth, full and productive employment and decent work for all

Goal 9. Build resilient infrastructure, promote inclusive and sustainable industrialization and foster innovation

Goal 10. Reduce inequality within and among countries

Goal 11. Make cities and human settlements inclusive, safe, resilient, and sustainable

Goal 12. Ensure sustainable consumption and production patterns

Goal 13. Take urgent action to combat climate change and its impacts

Goal 14. Conserve and sustainably use the oceans, seas, and marine resources for sustainable development

Goal 15. Protect, restore, and promote sustainable use of terrestrial ecosystems, sustainably manage forests, combat desertification, and halt and reverse land degradation and halt biodiversity loss

Goal 16. Promote peaceful and inclusive societies for sustainable development, private access to justice for all and build effective, accountable, and inclusive institutions at all levels

Goal 17. Strengthen the means of implementation and revitalize the Global Partnership for Sustainable Development

So far, we have determined how well an OECD country is achieving the Interpol goals with respect to the SDGs pertinent to them. We found that the Scandinavian countries were at the top in the achievement. In combating terrorism in general, we found that the Scandinavian countries were also at the top. We also show that the fuzzy similarity measure of the ranking of countries used by the SDGs relevant to terrorism compared with SDG 16 alone was high.

4.8 GTI index

We next delete those countries of the GTI Index that were not included or had no impact from terrorism. We then reverse rank the countries of the GTI rank so that the rankings would be compatible with that of the IPD average ranking (Table 4.6).

Table 4.6 GTI index.

Country	GTI	IPG Average
Australia	5	14
Austria	10	5
Belgium	12	6
Canada	14	12
Chile	25	19
Denmark	1	1
France	20	8
Germany	19	4
Greece	21	23
Ireland	2	16
Israel	23	22
Italy	15	18
Japan	9	10
Lithuania	3	21
Mexico	4	25
Netherlands	6	7
New Zealand	17	11
Norway	16	3
Slovak Rep.	13	20
Spain	11	15
Sweden	8	2
Switzerland	7	9
Turkey	24	24
United Kingdom	18	13
United States	22	17

Let A denote the GTI ranking and B the IPG ranking. Here $n = 25$. We find that $M(\mu_A, \mu_B) = \frac{243}{650-243} = 0.597$. From Section 4.4, we have that the smallest M can be is $\frac{n+1}{3n-1} = \frac{26}{74} = 0.351$. Thus $\frac{0.597-0.351}{1-0.351} = \frac{0.246}{0.649} = 0.379$. Also, $S(\mu_A, \mu_B) = \frac{2(0.597)}{1.597} = 0.748$. The smallest S can be is $\frac{2(0..351)}{1.351} = 0.520$. Hence $\frac{0.748-0.520}{1-0.520} = 0.475$.

See Exercise 7.

4.9 Bioterrorism

In this section we consider a model developed in [48]. Bioterrorism is defined by the Centers for Disease Control and Prevention (CDC) as the deliberate release of viruses, bacteria, or other germs (agents) used to cause illness or death in people, animals, or plants. In [48], a method of assessing how vulnerable a country is to a bioterrorism attacks is developed. In [5], a model is proposed for assessing a bioterrorist attack using the following equation:

$$\text{Bioterorist threat} = (\text{consequences of attack}) \times (\text{likelihood of attack}),$$

where

$$\begin{gathered}\text{consequence of attack} = (\text{value of asset being defended}) \times (\text{hazard posed by}\\ \text{agents}) \times (\text{vulnerability of assets being defended})\\ \text{and}\\ \text{likelihood of attack} = (\text{motivation}) \times (\text{capability of attackers}).\end{gathered}$$

The model in [48] is based somewhat on this equation, but not entirely. The three components of the equation for assessment of consequences of attacks are now explained and compared to factors used in this model.

Value of the assets being defended: This is a political decision that can be aided by economic estimates of the monetary worth assets.

Hazard posed by the agents: A variety of factors has to be considered These include level of communicability, average incubation period, and average rate of morality, among others. In [48], the average duration of the disease is also considered.

Vulnerability of the assets being defended: Factors such as concentration of populations in urban centers and the effects of globalization are considered [5]. The effect of globalization again reinforces the importance of considering the transmission rate of an infectious agent in this assessment. Values provided by the CIA World Factbook or urbanization and population residing in main cities are used since urban centers can be defined in a variety of ways.

Infectious agents

The U. S. department of Homeland security (DHS) has created a list of biological agents which are deemed the most important pathogens These pathogens are known as material threats. This list is comprised of various bacteria, viruses, and toxins of which the most commonly known and most plausible for use are smallpox (variola major), anthrax (*Bacillus anthraxes*), Ebola (a viral hemorrhagic fever), plague (*Yersinia pestis*), and botulism (*Clostrdium botulinum*) (www.nytimes.com). These five infectious agents re listed as high priority Category A agents by the CDC.

Factors in determining a country's vulnerability to a bioterrorism attack

The following factors will be considered when assessing how vulnerable a country is to a biological attack:

1. Urbanization
2. Percentage of population under 14 years old and percentage of population over 65 years old
3. Percentage of population residing in the largest cities
4. Physician density
5. Hospital bed density
6. Incubation period of the agent
7. Lethality of the agent
8. Duration of the disease
9. Degree of person-to-person transmission of the infectious agent

A list of assumptions made for the creation of the model can be found in [48, p. 67].

Equation behind determining a country' vulnerability to a bioterrorism attack

Using statistics from the CIA World Factbook, a score was determined for the top fifty most populated countries. This score is comprised of values of the nine factors mentioned previously. The scores for each country was determined using the following equation:

$$\text{Score} = (\text{pop})(\text{lethality})(\text{trans})(\text{treat}) + (\frac{\text{bed dens}}{\text{phys dens}})(\text{dur}) + (\text{inc}),$$

where

pop = (%urb)((%<14)+%>65)) + ($\sum$%maj cities)
lethality = percent lethality if disease is untreated
trans = availability of treatment
bed dens = number of hospital beds per 1000 people
phys dens = number of physicians per 1000 people
dur = duration of disease
inc = incubation period of disease
%urb = percent of population that is urbanized
%<14 = percent f population under 14 years old

%>65 = percent of population over 65

$\sum$%maj cities = percentage of population living in major cities

An explanation of the values used in the above equation can be found in [48, p. 68].

Mathematical calculations involving expert opinions

The above equation is applied to each of the five diseases, resulting in five new cores for each country. Through an electronic survey, a group of experts was asked to rank the severity of each of the five diseases based n how devastating the infectious agent could be on a population under the above assumptions. A copy of the survey can be found in [48].

The following matrix presents the results. The experts ranked the severity of each disease as "extremely dangerous," "moderately dangerous," "dangerous," "slightly dangerous," and "not dangerous." These rankings were converted into numerical values based on the following key: 5 = extremely dangerous, 4 = moderately dangerous, 3 = dangerous, 2 = slightly dangerous, 1 = not dangerous. We let E_i be expert i, where $i = 1, 2, \ldots, 7$ and F_1 = small pox, F_2 = anthrax, F_3 = Ebola, F_4 = plague, and F_5 = botulism (Table 4.7).

Table 4.7 Expert opinion.

	E_1	E_2	E_3	E_4	E_5	E_6	E_7
F_1	5	5	3	5	4	4	5
F_2	3	4	2	3	5	4	4
F_3	5	5	3	5	5	5	4
F_4	3	5	2	4	4	5	4
F_5	3	3	2	2	3	4	2

This matrix was then adjusted to an equivalent matrix by dividing all values by 5, which resulted in the matrix below.

$$A = \begin{array}{cccccccc} & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 \\ F_1 & 1.0 & 1.0 & 0.6 & 1.0 & 0.8 & 0.8 & 1.0 \\ F_2 & 0.6 & 0.8 & 0.4 & 0.6 & 1.0 & 0.8 & 0.8 \\ F_3 & 1.0 & 1.0 & 0.6 & 1.0 & 1.0 & 1.0 & 0.8 \\ F_4 & 0.6 & 1.0 & 0.4 & 0.8 & 0.8 & 1.0 & 0.8 \\ F_5 & 0.6 & 0.6 & 0.4 & 0.4 & 0.6 & 0.8 & 0.4 \end{array}$$

From the matrix A, we determine next the AHP, Guiasu, and Yen equations. The method of determining these equations can be found in Chapter 1.

AHP: $G = 0.231F_1 + 0.187F_2 + 0.239F_3 + 0.201F_4 + 0.148F_5$

Guiasu: $G = 0,234F_1 + 0.185F_2 + 0.0.240F_3 + 0.199F_4 + 0.142F_5$

Yen: $G = 0.232F_1 + 0.185F_2 + 0.239F_3 + 0.200F_4 + 0.143F_5$

Table 4.8 Bio and GTI.

Country	Bio Rank	GTI Rank	Country	Bio Rank	GTI Rank
China			South Africa	36	40
India	3	7	Spain	17	32
US	14	18	Columbia	26	9
Indonesia	12	16	Ukraine	29	38
Brazil	25	42	Tanzania	44	23
Pakistan	2	2	Kenya	41	12
Nigeria	11	4	Argentina	34	31
Bangladesh	6	26	Poland		
Russia	21	27	Uganda	19	28
Japan	40	32	Algeria	8	22
Mexico	15	37	Canada	33	29
Philippines	1	11	Sudan		
Ethiopia	38	30	Morocco	23	41
Vietnam	13	43	Iraq	32	3
Egypt	4	10	Afghanistan	9	1
Germany	24	20	Nepal	42	21
Turkey	28	15	Peru	30	24
Iran	18	13	Malaysia	20	39
Congo	39	8	Uzbekistan	7	36
Thailand	37	17	Venezuela	27	44
France	31	19	Saudi Arabia	35	34
UK	22	25	Ghana		
Italy	10	35	Yemen	16	14
Burma	5	5	North Korea		
South Korea			Mozambique	43	6

The rankings by the AHP, Guiasu, and Yen methods were exactly the same. We next provide the ranking together with the GTI rank so that we can provide the fuzzy similarity measure of the two rankings. We do not include the countries of the GTI ranking which had no impact or were not included In order for the rankings to be compatible, i.e., small numbers in the rankings represent high impact and high risk, we reverse order the bioterrorism rankings (Table 4.8).

Let A denote the Bio ranking, i.e., the first column and let B denote the ranking, i.e., the second column. Here $n = 44$. Now $M(\mu_A, \mu_B) = \frac{704}{1980-704} = \frac{704}{1276} = 0.552$. The smallest M can be is $\frac{n+2}{3n+2} = 0.343$. Hence $\frac{0.552-0.343}{1-0.343} = \frac{0.209}{0.657} = 0.318$. Now $S(\mu_A, \mu_B) = \frac{2M(\mu_A,\mu_B)}{1+M(\mu_A,\mu_B)} = \frac{2(0.552)}{1.552} = \frac{1.104}{1.552} = 0.711$. The smallest S can be is $\frac{2(0.343)}{1.343} = \frac{0.686}{1.343} = 0.511$. Thus $\frac{0.711-.511}{1-0.511} = \frac{0.200}{0.489} = 0.408$.

We next consider preference relations. We use the values from matrix A. Let $R_k = [\rho_{ij}^k]$, where $\rho_{ij}^k(F_i, F_j) = (e_{ik} - e_{jk} + 0.5) \wedge 1$ if $e_{ik} \geq e_{jk}$ and $\rho_{ij}^k(F_i, F_j) =$

$1-(e_{jk}-e_{ik}+0.5)\wedge 1$ if $e_{ik}<e_{jk}$ $\rho_{ij}^k(F_i,F_j)=\{(e_{ik}-e_{jk}+0.5)\wedge 1$ if $e_{ik}\geq e_{jk}$, where $i, j=1,2,3,4,5$ and $k=1,...,7$. Then

$R_1 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.9	0.5	0.9	0.9
F_2	0.1	0.5	0.1	0.5	0.5
F_3	0.5	0.9	0.5	0.9	0.9
F_4	0.1	0.5	0.1	0.5	0.5
F_5	0.1	0.5	0.1	0.5	0.5

and $R_2 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.7	0.5	0.5	0.9
F_2	0.3	0.5	0.3	0.3	0.7
F_3	0.5	0.7	0.5	0.5	0.9
F_4	0.5	0.7	0.5	0.5	0.9
F_5	0.1	0.3	0.1	0.1	0.5

$R_3 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.7	0.5	0.7	0.7
F_2	0.3	0.5	0.3	0.5	0.5
F_3	0.5	0.7	0.5	0.7	0.7
F_4	0.3	0.5	0.3	0.5	0.5
F_5	0.3	0.5	0.3	0.5	0.5

and $R_4 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.9	0.5	0.7	1.0
F_2	0.1	0.5	0.1	0.3	0.7
F_3	0.5	0.9	0.5	0.7	1.0
F_4	0.3	0.7	0.3	0.5	0.9
F_5	0.0	0.3	0.0	0.1	0.5

$R_5 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.3	0.3	0.5	0.7
F_2	0.7	0.5	0.5	0.7	0.9
F_3	0.7	0.5	0.5	0.7	0.9
F_4	0.5	0.3	0.3	0.5	0.7
F_5	0.3	0.1	0.1	0.3	0.5

and $R_6 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.5	0.3	0.3	0.5
F_2	0.5	0.5	0.3	0.3	0.5
F_3	0.7	0.7	0.5	0.5	0.7
F_4	0.7	0.7	0.5	0.5	0.7
F_5	0.5	0.5	0.3	0.3	0.5

$R_7 =$

	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.7	0.7	0.7	1.0
F_2	0.3	0.5	0.5	0.5	0.9
F_3	0.3	0.5	0.5	0.5	0.9
F_4	0.3	0.5	0.5	0.5	0.9
F_5	0.0	0.1	0.1	0.1	0.5

Let $A_k=[a_{ij}^k]$, where $a_{ij}^k=1$ if $\rho_{ij}^k>0.5$ and 0 otherwise, $i.j=1,2,3,4,5$ and $k=1,...,7$.

Let $R=[r_{ij}]$, where for all $i, j=1,2,3,4,5$, $r_{ij}=\frac{1}{7}\sum_{k=1}^{7}a_{ij}^k$ if $i\neq j$ and 0 otherwise. Then

$R =$

	F_1	F_2	F_3	F_4	F_5
F_1	0	5/7	1/7	4/7	6/7
F_2	1/7	0	0	1/7	4/7
F_3	2/7	5/7	0	1/7	1
F_4	1/7	3/7	0	0	5/7
F_5	0	0	0	0	0

Let $G = [g_{ij}]$, where for all $i, j = 1, 2, 3, 4, 5$, $g_{ij} = 1$ if $r_{ij} > 0.5$ and 0 otherwise. Then g_{ij} expresses whether F_i defeats F_j. Then

$G =$

	F_1	F_2	F_3	F_4	F_5
F_1	0	1	0	1	1
F_2	0	0	0	0	1
F_3	0	1	0	1	1
F_4	0	0	0	0	1
F_5	0	0	0	0	0

Now $g_i = \frac{1}{4}\sum_{j=1}^{5} g_{ij}$ is the mean degree to which F_i is preferred to all other F_j. We find that $g_1 = 0.75$, $g_2 0.25$, $g_3 = 0.75$, $g_4 0.25$, and $g_5 = 0$.

Let z_Q^i be the fuzzy consensus winner or the extent to which F_i is preferred to Q other F_j, where $i, j = 1, 2, 3, 4, 5$ and Q denotes most. We define $z_Q^i = \mu_Q(g_i) = 1$ if $0.8 \le g_i \le 1$, $2g_i - 0.6$ if $0.3 < g_i < 0.8$, and 0 if $0 \le g_i \le 0.3$. Then

$$z_Q^1 = \mu_Q(g_1) = \mu_Q(\frac{3}{4}) = 0.9$$
$$z_Q^2 = \mu_Q(g_2) = \mu_Q(\frac{1}{4}) = 0.0$$
$$z_Q^3 = \mu_Q(g_3) = \mu_Q(\frac{3}{4}) = 0.9$$
$$z_Q^4 = \mu_Q(g_4) = \mu_Q(\frac{1}{4}) = 0.0$$
$$z_Q^5 = \mu_Q(g_5) = \mu_Q(0) = 0.0.$$

From these values, we see that F_2 is not preferred to any other F_i, where $i = 1, 3, 4, 5$. Similarly, F_4 is not preferred to any other F_i, where $i = 1, 2, 3, 5$, and F_5 is not preferred to any other F_i, where $i = 1, 2, 3, 4$. Also, F_1 is preferred greatly to all other F_i, where $i = 2, 3, 4, 5$ and F_3 is preferred greatly to any other F_i, where $i = 1, 2, 4, 5$. From this information, it can be concluded that the experts polled believe using either smallpox or Ebola in a bioterrorism attack would be the most beneficial to the attacking party because either of these diseases would cause more damage than anthrax, plague, or botulism. This conclusion correlates with information about the disease. Both smallpox and Ebola are biosafety 4 agents whereas anthrax, plague, and botulism are all biosafety level 2 agents.

Fuzzy preference relations were used further to determine relationships between each of the diseases.

Let $S, I, >, R, \sim$ be relations on X having the following meanings:

S: outranking relation xSy means x is not worse than y

I: indifference relation xLy means x and y are indifferent

R: incompatibility relation xRy means x and y are incomparable

$>$: preference relation $x > y$ means x is preferred to y

$\sim$: non-preference relation $x \sim y$ means that x and y cannot be discriminated between

More specifically,
S determines the degree to which one disease is not worse than another. The larger the value of xSy, the more similar two diseases are in terms of impact on a population. Conversely, the smaller the value of xSy, more dissimilar two diseases are in terms of impact on a population.

I determines the degree to which two diseases are indifferent to each other. The values determined from this matrix are somewhat irrelevant in analysis of a country's vulnerability to bioterrorism attack. The indifference relation is used in analysis of factors that are interrelated. In the case of bioterrorism vulnerability, the release of one biological agent is not related to the release of another biological agent. If the decision to release a given infectious agent, however, was dependent on the release of another agent, this preference relation would indicate the degree of the release of a given infectious agent was independent of the release of another infectious agent.

R determines the degree to which two diseases are incomparable. Similar to the indifference relation I, the values determined from this matrix are somewhat irrelevant in analysis of a country's vulnerability to a bioterrorism attack. The incompatibility relation is used in analysis of related factors and when two factors are compared. Since the release of one biological agent, this comparison is not helpful in our analysis. I, however, the release was one disease was related to the release of another disease, the incompatibility preference relation would indicate the degree to which two infectious agents cannot be compared.

$>$ determines the degree to which one disease is preferred over another. The larger the value $x > y$ the more disease x is preferred over disease y for use in a bioterrorism attack. Conversely, the smaller the value $x > y$, the less disease x is preferred over disease for use in a bioterrorism attack.

$\sim$ determines the degree to which discrimination between two diseases is possible. The larger the value $x \sim y$, the more one can tell the difference in the effects of a bioterrorism attacks utilizing a given infectious agent. Conversely, the smaller $x \sim y$, the less one can tell the difference in the effects of a bioterrorism attack utilizing a given infectious agent.

We now define the following indices from $S : \forall x, y \epsilon X$,

Indifference index $I(S)(x, y) = S(x, y) \wedge S(y, x)$

Incompatibility index $R(S)(x,) = (1 - S(x, y)) \wedge (1 - S(y, x))$

Preference index $> (S)(x, y) = S(x, y) \wedge (1 - S(y, x))$

Non-preference index $\sim (S)(x, y) = (S(x, y) \wedge S(y, x)) \vee ((1 - S(x, y)) \wedge (1 - S(y, x)))$.

Let $S = R$, where $R =$

	F_1	F_2	F_3	F_4	F_5
F_1	0	5/7	1/7	4/7	6/7
F_2	1/7	0	0	1/7	4/7
F_3	2/7	5/7	0	4/7	1
F_4	1/7	3/7	0	0	5/7
F_5	0	0	0	0	0

Then

$I(S) =$

	F_1	F_2	F_3	F_4	F_5
F_1	0	1/7	1/7	1/7	0
F_2	1/7	0	0	1/7	0
F_3	1/7	0	0	0	0
F_4	1/7	1/7	0	0	0
F_5	0	0	0	0	0

$R(S) =$

	F_1	F_2	F_3	F_4	F_5
F_1	1	2/7	5/7	3/7	1/7
F_2	2/7	1	2/7	4/7	3/7
F_3	5/7	2/7	1	3/7	0
F_4	3/7	4/7	3/7	1	2/7
F_5	1/7	3/7	0	2/7	1

$> (S) =$

	F_1	F_2	F_3	F_4	F_5
F_1	0	5/7	1/7	4/7	6/7
F_2	1/7	0	0	1/7	4/7
F_3	2/7	5/7	0	4/7	1
F_4	1/7	3/7	0	0	5/7
F_5	0	0	0	0	0

$\sim (S) =$

	F_1	F_2	F_3	F_4	F_5
F_1	1	2/7	5/7	3/7	1/7
F_2	2/7	1	2/7	4/7	3/7
F_3	5/7	2/7	1	3/7	0
F_4	3/7	4/7	3/7	1	2/7
F_5	1/7	3/7	0	2/7	1

For the outranking relation S, the matrix can be interpreted as follows:
F_1 is not worse than F_2 with intensity 5/7

For the indifference relation I, the matrix can be interpreted as follows:
F_1 and F_2 are indifferent with intensity 1/7

For the incompatibility relation R, the matrix can be interpreted as follows:
F_1 and F_2 are incomparable with intensity 2/7

For the preference relation $>$, the matrix can be interpreted as follows:
F_1 is preferred to F_2 with intensity $5/7$

For the non-preference relation $\sim$, the matrix can be interpreted as follows:
F_1 and F_2 can be discriminated between with intensity $2/7$

4.10 Exercises

1. Complete the interpretations given immediately above.

We now consider the degree of agreement between the experts $m, n = 1, ..., 7$, $\nu(m,n) = 1$ if $|\rho_{ij}^m - \rho_{ij}^n| \leq 1 - 0.94 = 0.06$ and 0 otherwise.

2. Find the $\nu(m,n)$. For example,

$\nu(1,2) =$

	F_1	F_2	F_3	F_4	F_5
F_1	1	0	1	0	1
F_2	0	1	0	0	0
F_3	1	0	1	0	1
F_4	0	0	0	1	0
F_5	1	0	1	0	1

The degree of agreement between the experts m, n as to their preference is determined by $\nu_B(m,n) = \frac{1}{10}\sum_{i=1}^{4}\sum_{j=i+1}^{5}\nu_{ij}(m,n)$.

3. Find the $\nu_B(m,n)$. For example, $\nu_B(1,2) = \frac{3}{10}$.

The degree of agreement of all pairs of experts m, n is given by the formula

$$\nu^B = \frac{1}{21}\sum_{m=1}^{6}\sum_{n=m+1}^{7}\nu_B(m,n).$$

4. Show that $\nu^B = \frac{1}{21}(\frac{52}{10}) = \frac{26}{105}$.

From this, we conclude that on average all of the experts agree on the severity of the disease about 23% of the time. The measure, however, only determines to what degree the experts agree exactly on the severity of a disease based on the previously calculated preference relations. Similarly, we can consider the degree of agreement between experts under different constraints so that the range of differences in opinions is greater. When adjusting the expert opinions to be between 0 and 1 rather than 1 and 5, the smallest possible difference between the expert rankings is 0.2. Translated to the R_i matrices, where $i = 1, 2, ..., 7$, which are used when calculating the $\nu(i,j)$ matrices. This means that the smallest degree of agreement between the experts is 0.2. We now consider the degree of agreement between the experts $m, n = 1, ..., 7$, where $\nu(m,n) = 1$ if $|\rho_{ij}^m - \rho_{ij}^n| \leq 1 - 0.80 = 0.20$ and 0 otherwise.

5. Calculate the $\nu(m, n)$ under this definition. For example,

$\nu(1, 2) =$

	F_1	F_2	F_3	F_4	F_5
F_1	1	1	1	0	1
F_2	1	1	1	1	1
F_3	1	1	1	0	1
F_4	0	1	0	1	0
F_5	1	1	1	0	1

6. Calculate the $\nu_B(m, n)$ and ν^B. Show that $\nu^B = \frac{27}{35}$.

From this result, conclude that the experts agree within 1 ranking point about 77% of the time. This means that the experts ranked each disease with similar severity, indicating that the analysis using expert opinions is somewhat valid because there are no outliers in ranking values.

7. Let A' denote the ranking of the complement of the A ranking of Table 4.6. That is, if $\mu_A(x) = a$, then $\mu_{A'}(x) = 1 - a$. Show that $(M(\mu_{A'}, \mu_B) - 0.351)/(1 - 0.352) = 0.139$.

CHAPTER

5 Fuzzy implication operators: health security and political risk[*]

This chapter rests heavily on [76]. The Global Health Security Index states that all countries remain dangerously unprepared for future epidemic and pandemic threats, including threats potentially more devastating than COVID-19, [39]. In this chapter, we rank the Organization for Economic Cooperation and Development (OECD) countries with respect to their preparation. In [25], countries are ranked with respect to their health care. We find the fuzzy similarity measure between these two rankings. We do this in a manner not previously used. We use implication operators to define a new fuzzy similarity measure to find the fuzzy similarity of these rankings. We also consider the natural disaster risk, the political stability, and the political risk of OECD countries. We provide the rankings as given in [61,87,88]. The report in [61] systematically considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. In each case, we found the similarities to be medium.

5.1 Preliminary results

Let μ, ν be fuzzy subsets of a set X. Then recall that M and S are fuzzy similarity measures on $\mathcal{FP}(X)$, where

$$M(\mu,\nu)=\frac{\sum_{x\in X}\mu(x)\wedge\nu(x)}{\sum_{x\in X}\mu(x)\vee\nu(x)},$$

$$S(\mu,\nu)=1-\frac{\sum_{x\in X}|\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))}.$$

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00013-1

Definition 5.1.1. [9, p. 14] Let I be a function of $[0,1] \times [0,1]$ into $[0,1]$ such that $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. Then I is called an **implication operator**.

An implication operator I is said to satisfy the **identity principle** if $I(x,x) = 1$ for all $x \in [0,1]$. An implication operator is said to satisfy the **ordering principle** if $x \le y \Leftrightarrow I(x,y) = 1$. Clearly, the ordering principle implies the identity principle.

I_1, I_2, and L defined below are implication operators that satisfy the ordering principle.

Example 5.1.2. Let $x, y \in [0,1]$.

(1) Godel implication operator: $I_1(x,y) = 1$ if $x \le y$, $I_1(x,y) = y$ otherwise.

(2) Goguen implication operator: $I_2(x,y) = 1$ if $x \le y$ and $I_2(x,y) = y/x$ otherwise

(3) Luckasiewicz implication operator: $L(x,y) = (1 - x + y) \wedge 1$.

Other implication operators can be found in [9].

Let X be a set with n elements, $n > 1$, say $X = \{x_1, ..., x_n)$. Let A be one-to-one function of X onto $\{1, ..., n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X by for all $x \in X$, $\mu_A(x) = \frac{A(x)}{n}$. Then μ_A is called the **fuzzy subset associated with** A.

Example 5.1.3. (1) Let $X = \{x_1, ..., x_6\}$. Define A and B as follows: $A(x_1) = 2$, $A(x_2) = 1$, and $A(x_i) = i$, $i = 3,4,5,6$. Define $B(x_i) = i$, $i = 1, ..., 6$. Then $I_1(\mu_A(x_1), \mu_B(x_1)) = \frac{1}{6}$ and $I_1(\mu_B(x_2), \mu_A(x_2) = \frac{1}{6}$. Also, $I_2(\mu_A(x_1), \mu_B(x_1)) = \frac{1}{2}$, and $I_2(\mu_B(x_2), \mu_A(x_2) = \frac{1}{2}$. Now $L(\mu_A(x_1), \mu_B(x_1)) = 1 - \frac{2}{6} + \frac{1}{6} = 1 - \frac{1}{6} = \frac{5}{6}$ and $L(\mu_B(x_2), \mu_A(x_2)) = \frac{5}{6}$.

(2) Let $X = \{x_1, ..., x_6\}$. Define A and B as follows: $A(x_5) = 6$, $A(x_6) = 5$, and $A(x_i) = 1$, $i = 1,2,3,4$. Define $B(x_i) = i$, $i = 1, ..., 6$. Then $I_1(\mu_A(x_5), \mu_B(x_5)) = \frac{5}{6}$ and $I_1(\mu_B(x_6), \mu_A(x_6) = \frac{5}{6}$. Also, $I_2(\mu_A(x_6), \mu_B(x_6)) = \frac{5}{6}$ and $I_2(\mu_B(x_5), \mu_A(x_5) = \frac{5}{6}$. Now $L(\mu_A(x_5), \mu_B(x_5)) = 1 - \frac{6}{6} + \frac{5}{6} = 1 - \frac{1}{6} = \frac{5}{6}$ and $L(\mu_B(x_6), \mu_A(x_6)) = \frac{5}{6}$.

Definition 5.1.4. [9, p. 15] Let I be an implication operator. Define the fuzzy subset E_I of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by for all $\mu_A, \mu_B \in \mathcal{FP}(X)$,

$$E_I(\mu_A, \mu_B) = \wedge\{\wedge\{I(\mu_A(x), \mu_B(x)) | x \in X\}, \wedge\{I(\mu_B(x), \mu_A(x)) | x \in X\}\}.$$

Then $E_I(\mu_A, \mu_B)$ is called the **degree of sameness** of μ_A and μ_B.

5.2 Main results

Unless some importance is given for the various orders, it seems that L of Example 5.1.2 is the best choice for an implication operator for rankings.

Let L be the Luckasiewicz implication operator. Then $L(a,b) \wedge L(b,a) = (1-a) \wedge (1-b) + a \wedge b$ by [122]. Thus $L(a,b) \wedge L(b,a) = 1 - a \vee b + a \wedge b$.

Theorem 5.2.1. *Define* $S_L(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\in X}[(1-\mu_A(x)) \wedge (1-\mu_B(x)) + \mu_A(x) \wedge \mu_B(x)]$*, where* X *is set with* n *elements. Then* S_L *is a fuzzy similarity measure.*

Proof. Clearly, $S_L(\mu_A, \mu_B) = S_L(\mu_B, \mu_A)$. Now

$$\begin{aligned} S_L(\mu_A, \mu_B) = 1 &\Leftrightarrow \frac{1}{n}\sum_{x\in X}[(1-\mu_A(x)) \wedge (1-\mu_B(x)) + \mu_A(x) \wedge \mu_B(x)] = 1 \\ &\Leftrightarrow \frac{1}{n}\sum_{x\in X}[(1-\mu_A(x) \vee \mu_B(x) + \mu_A(x) \wedge \mu_B(x)] = 1 \\ &\Leftrightarrow \sum_{x\in X}[(1-\mu_A(x) \vee \mu_B(x) + \mu_A(x) \wedge \mu_B(x)] = n \\ &\Leftrightarrow 1-\mu_A(x) \vee \mu_B(x) + \mu_A(x) \wedge \mu_B(x) = 1 \forall x \in X \\ &\Leftrightarrow \mu_A = \mu_B. \end{aligned}$$

Suppose $\mu_A \subseteq \mu_B \subseteq \mu_C$. Then

$$\begin{aligned} S_L(\mu_A, \mu_C) &= \sum_{x\in X}[1 + \mu_A(x) \wedge \mu_C(x) - \mu_A(x) \vee \mu_C(x)] \\ &= n + \sum_{x\in X}[\mu_A(x) - \mu_C(x)], \\ S_L(\mu_A, \mu_B) &= \sum_{x\in X}[1 + \mu_A(x) \wedge \mu_B(x) - \mu_A(x) \vee \mu_B(x)] \\ &= n + \sum_{x\in X}[\mu_A(x) - \mu_B(x], \\ S_L(\mu_B, \mu_C) &= \sum_{x\in X} 1 + [\mu_B(x) \wedge \mu_C(x) - \mu_B(x) \vee \mu_C(x)] \\ &= n + \sum_{x\in X}[\mu_B(x) - \mu_C(x)]. \end{aligned}$$

Now $\forall x \in X$, $\mu_A(x) - \mu_C(x) \leq (\mu_A(x) - \mu_B(x)) \wedge (\mu_B(x) - \mu_C(x))$. Hence $S_L(\mu_A, \mu_C) \subseteq S_L(\mu_A, \mu_B) \wedge S_L(\mu_B, \mu_C)$.

Suppose $S_L(\mu_A, \mu_B) = 0$. Then $\frac{1}{n}\sum_{x\in X}[(1-\mu_A(x)) \wedge (1-\mu_B(x)) + \mu_A(x) \wedge \mu_B(x)] = 0$. Thus $\forall x \in X$, $(1-\mu_A(x)) \wedge (1-\mu_B(x)) + \mu_A(x) \wedge \mu_B(x) = 0$. Hence $\forall x \in X$, $\mu_A(x) \wedge \mu_B(x) = 0$. ■

We have shown that S_L is a fuzzy similarity measure. S_L was induced by the Luckasiewicz implication operator L.

We next consider a different definition of sameness.

Example 5.2.2. Let $n = 6$ and $X = \{x_i | i = 1, ..., 6\}$. Define $A, B : X \to [0, 1]$ as follows: $A(x_i) = n - i + 1$ $B(x_i) = 1$ for $i = 1, ..., 6$. Then $L(\mu_A, \mu_B) = \frac{1}{6}[(\frac{1}{6}(5 + 3 + 1) + 1 + 1 + 1] = \frac{1}{6}[\frac{9}{6} + 3] = 0.75$. By the symmetry of the situation, $L(\mu_B, \mu_A) = 0.75$. Hence $E_L = 0.75 \wedge 0.75 = 0.75$. However, the sameness $\frac{1}{6}[\frac{9}{6} + \frac{9}{6}] = 0.5$ seems more reasonable in this case.

Definition 5.2.3. Let I be an implication operator. Define $S : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0, 1]$ by for all $(\mu, \nu) \in \mathcal{FP}(X) \times \mathcal{FP}(X)$, $S(\mu, \nu) = \frac{1}{n} \sum_{x \in X} I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x)))$. Then S is called a **degree of likeness**.

An implication operator I is called a **hybrid monotonous implication operator** if $I(x, \_)$ is non decreasing for all $x \in [0, 1]$ and $I(\_, y)$ is nonincreasing for all $y \in [0, 1]$.

Theorem 5.2.4. *Suppose I is a hybrid monotonous implication operator that satisfies the ordering principle. Then S of Definition 5.2.3 is a fuzzy similarity measure.*

Proof. Clearly $S(\mu, \nu) = S(\nu, \mu)$.

Now $S(\mu, \nu) = 1 \Leftrightarrow \sum_{x \in X} I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x))) = n \Leftrightarrow I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x))) = 1$ for all $x \in X \Leftrightarrow I((\mu(x), \nu(x)) = 1$ and $I((\nu(x), \mu(x))) = 1$ for all $x \in X \Leftrightarrow \mu(x) = \nu(x)$ for all $x \in X \Leftrightarrow \mu = \nu$.

Suppose $\mu \subseteq \nu \subseteq \rho$. Then for all $x \in X$,

$$I(\mu(x), \nu(x)) \leq I(\mu(x), \rho(x)) \text{ and } I(\rho(x), \mu(x)) \leq I(\nu(x), \mu(x)),$$
$$I(\mu(x), \nu(x)) \leq I(\nu(x), \rho(x)) \text{ and } I(\rho(x), \mu(x)) \leq I(\rho(x).\nu(x)).$$

Thus for all $x \in X$,

$$\begin{aligned} I(\mu(x), \rho(x)) \wedge I(\rho(x), \mu(x)) &= I(\rho(x), \mu(x)) \\ &\leq I(\nu(x), \mu(x)) = I(\mu(x), \nu(x)) \wedge I(\nu(x), \mu(x)) \end{aligned}$$

and

$$\begin{aligned} I(\mu(x), \rho(x)) \wedge I(\rho(x), \mu(x)) &= I(\rho(x), \mu(x)) \\ &\leq I(\rho(x), \nu(x)) = I(\rho(x), \nu(x)) \wedge I(\nu(x), \rho(x)). \end{aligned}$$

Therefore for all $x \in X$,

$$\begin{aligned} &I(\mu(x), \rho(x)) \wedge I(\rho(x), \mu(x)) \\ &\leq [I(\mu(x), \nu(x)) \wedge I(\nu(x), \mu(x))] \wedge [I(\rho(x), \nu(x)) \wedge I(\nu(x), \rho(x))]. \end{aligned}$$

Hence $S(\mu, \rho) \leq S(\mu, \nu) \wedge S(\nu, \rho)$.

Suppose $S(\mu, \nu) = 0$. Then $\sum_{x \in X} I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x))) = 0$ and so $I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x))) = 0$ for all $x \in X$. Hence $I((\mu(x), \nu(x)) = 0$ or $I((\nu(x), \mu(x))) = 0$ for all $x \in X$. Hence $\mu(x) \wedge \nu(x) = 0$ for all $x \in X$. ■

Using the new definition of sameness (likeness), we get

$S_1 : S_1(\mu,\nu) = \frac{1}{n}\sum_{x\in X}[I_1(\mu(x),\nu(x)) \wedge I_1(\nu(x),\mu(x))$. Define $\bar{\wedge} : [0,1] \times [0,1] \to [0,1]$ by $\bar{\wedge}(a,b) = 1$ if $a = b$ and $a \wedge b$ if $a \neq b$. Then $S_1(\mu,\nu) = \frac{1}{n}\sum_{x\in X}\mu(x)\,\bar{\wedge}\,\nu(x)$.

$S_2 : S_2(\mu,\nu) = \frac{1}{n}\sum_{x\in X}[I_2(\mu(x),\nu(x)) \wedge I_2(\nu(x),\mu(x))$. Define $\varnothing : [0,1] \times [0,1] \to [0,1]$ by $\varnothing(a,b) = 1$ if $a = b$ and $\frac{a\wedge B}{a\vee b}$ if $a \neq b$. Then $S_2(\mu,\nu) = \frac{1}{n}\sum_{x\in X}\mu(x)\varnothing\nu(x)$.

S_L: $S_L(\mu.\nu) = \frac{1}{n}\sum_{x\in X}[(1-\mu(x)+\nu(x))\wedge 1)] \wedge [(1-\nu(x)+\mu(x))\wedge 1]$

$= \frac{1}{n}\sum_{x\in X}[(1-\mu(x)+\nu(x))1)] \wedge [(1-\nu(x)+\mu(x))] = \frac{1}{n}\sum_{x\in X}(1-\mu(x)\vee \nu(x)+\mu(x)\wedge\nu(x))$

$= 1 - \frac{1}{n}\sum_{x\in X}(\mu(x)\vee\nu(x) - \mu(x)\wedge\nu(x))$.

Consider M of Example 5.1.2. Clearly the smallest value M can be with respect to rankings of a set with n elements is determined as follows: Consider the rank A defined by $A(x_i) = n-i+1, i = 1,...,n$. Consider the rank B defined by $B(x_i) = i$, $i = 1,...,n$. Then $\frac{\sum_{i=1}^{n}\mu_A(x_i)\wedge\mu_B(x_i)}{\sum_{i=1}^{n}\mu_A(x_i)\vee\mu_B(x_i)}$ has the smallest values in the numerator and largest values in the denominator. Thus the smallest value M can is determined by these rankings. The rankings of other fuzzy similarity measures can be determined by a formula relating them to M.

We next consider the smallest S_L can be under the new definition of sameness. Suppose n is even. (Note that then for rankings, $\mu(x) \neq \nu(x)$ for all $x \in X$.) Then $S_L(\mu_A,\mu_B) = \frac{1}{n}(n - 2\frac{1}{n}\frac{n}{2}\frac{n}{2}) = 1 - \frac{1}{2} = 0.5$. Suppose n is odd. Then $S_L(\mu_A,\mu_B) = \frac{1}{n}(n - 2\frac{1}{n}(\frac{n+1}{2}\frac{n-1}{2})) = 1 - 2\frac{n^2-1}{4n^2} = 1 - \frac{1}{2} + \frac{1}{2n^2} = 0.5 + \frac{1}{2n^2}$.

Proposition 5.2.5. $S_L = S + \frac{1}{n}(S-1)$ *for S of Example 5.1.2.*

Proof. We have $S(\mu_A,\mu_B) = 1 - \frac{\sum|\mu_A(x)-\mu_B(x)|}{\sum(\mu_A(x)+\mu_B(x))} = 1 - \frac{\sum|\mu_A(x)-\mu_B(x)|}{n+1}$ and $S_L(\mu_A,\mu_B) = 1 - \frac{1}{n}\sum(\mu(x)\vee\nu(x) - \mu(x)\wedge\nu(x)) = 1 - \frac{\sum|\mu_A(x)-\mu_B(x)|}{n}$. Hence $(S_L-1)n = (S-1)(n+1)$. Thus $nS_L = (n+1)S - 1$ or $nS_L = nS + S - 1$. Therefore $S_L = S + \frac{1}{n}(S-1)$. ■

We next consider the smallest S_L can be. Suppose n is even. Then $S = \frac{n/2+1}{n+1}$, [71]. Thus

$$\begin{aligned} S_L &= S + \frac{1}{n}(S-1) = S(1+\frac{1}{n}) - \frac{1}{n} \\ &= \frac{n/2+1}{n+1}(\frac{n+1}{n}) - \frac{1}{n} \\ &= \frac{1}{2} + \frac{1}{n} - \frac{1}{n} = \frac{1}{2}. \end{aligned}$$

Suppose n is odd. Then $S = \frac{1}{2} + \frac{1}{2n}$, [71]. Thus

$$\begin{aligned} S_L &= S(1+\frac{1}{n}) - \frac{1}{n} \\ &= (\frac{1}{2}+\frac{1}{2n})(1+\frac{1}{n}) - \frac{1}{n} \\ &= \frac{1}{2}+\frac{1}{2n}+\frac{1}{2n}+\frac{1}{2n^2}-\frac{1}{n} \\ &= \frac{1}{2}+\frac{1}{2n^2}. \end{aligned}$$

Theorem 5.2.6. *$M \subseteq S_L \subseteq S$ for M and S defined in Section 5.1, with $|X| \geq 2$.*

Proof. We first show that $S_L \subseteq S$. By Proposition 5.2.5, we have $S_L = S + \frac{1}{n}(S-1) \leq S$ since $S - 1 \leq 0$. (Also, $S_L \geq 0.5$ for $n \geq 2$.) We next show $M \subseteq S_L$.

$$\begin{aligned} M &= \frac{S}{2-S} = \frac{nS_L+1}{n+1}/(2-\frac{nS_L+1}{n+1}) = \frac{nS_L+1}{n+1}/\frac{2(n+1)-nS_L-1}{n+1} \\ &= \frac{nS_L+1}{2n+2-nS_L-1} = \frac{nS_L+1}{2n+1-nS_L}. \end{aligned}$$

We now show $\frac{nS_L+1}{2n+1-nS_L} \leq S_L$. We have $\frac{nS_L+1}{2n+1-nS_L} \leq S_L \Leftrightarrow nS_L + 1 \leq (2n+1-nS_L)S_L \Leftrightarrow 1 \leq nS_L + S_L - nS_L^2 \Leftrightarrow 1 \leq S_L + nS_L(1-S_L) \Leftrightarrow 1 \leq S_L + n(S_L - S_L^2)$. The latter inequality is true for $n \geq 2$ since $S_L \geq 0.5$. It suffices to show this follows for $n = 2$. Let $f(x) = x + 2(x - x^2)$. Then $f(\frac{1}{2}) = 1 = f(1)$. By elementary calculus, f takes a maximum at $x = \frac{3}{4}$. Now $f(\frac{3}{4}) = \frac{9}{8}$. ■

5.3 Security index

The 2021 Global Health Security Index measures the capacities of 195 countries to prepare for epidemics and pandemics. All countries remain dangerously unprepared for future epidemics and pandemic threats, including threats potentially more devastating than COVID-19, [39]. In Ref. [76], a ranking of countries with respect to health care is provided. We provide the ranking with respect to OECD countries (Table 5.1).

Let M and S be the fuzzy similarity measures defined in Section 5.1. We deleted the countries in the Health Security ranking that were not in the Health Care ranking and then reranked the Health Security countries. We found that $S(\mu_A, \mu_B) = 1 - \frac{223}{1122} = 1 - 0.199 = 0.801$. Hence $M(\mu_A, \mu_B) = \frac{S(\mu_A,\mu_B)}{2-S(\mu_A,\mu_B)} = \frac{0.801}{1.199} = 0.668$. With the countries deleted, $n = 33$. Thus the smallest M can be is $\frac{n+1}{3n-1} = \frac{34}{98} = 0.347$. The smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{66} = 0.515$. Therefore

$$\frac{0.668-0.347}{1-0.347} = \frac{0.321}{0.653} = 0.492$$

Table 5.1 Health.

Country	Health Security	Health Care	Country	Health Security	Health Care
Australia	2	9	Korea, Rep.	8	1
Austria	22	5	Latvia	14	
Belgium	19	4	Lithuania	18	26
Canada	4	19	Luxembourg	35	
Chile	23	30	Mexico	21	23
Czech Rep.	30	12	Netherlands	10	3
Denmark	11	8	New Zealand	12	16
Estonia	25	18	Norway	17	13
Finland	3	11	Poland	24	29
France	13	6	Portugal	27	22
Germany	7	10	Slovak Rep.	29	28
Greece	32	31	Slovenia	5	27
Hungary	28	33	Spain	15	7
Iceland	34		Sweden	9	20
Ireland	26	32	Switzerland	20	17
Israel	36	15	Turkey	33	21
Italy	31	25	United Kingdom	6	14
Japan	16	2	United States	1	24

and

$$\frac{0.801 - 0.515}{1 - 0.515} = \frac{0.286}{0.485} = 0.590.$$

We have by Proposition 5.2.5 that $S_L = S + \frac{1}{n}(S - 1)$. Thus $S_L(\mu_A, \mu_B) = 0.801 + \frac{1}{33}(0.801 - 1) = 0.801 - 0.006 = 0.795$. The smallest $S_L(\mu_A, \mu_B)$ is $\frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2} + \frac{1}{2178} = 0.5 + 0.000459$, which we round off to 0.5. Thus $\frac{0.795-0.5}{1-0.5} = 0.59$.

Converting a fuzzy similarity measures to a measure using the smallest value it can be, converts the measure to the interval [0, 1]. We can say if this converted value lies between 0 and 0.2, the similarity is very low, from 0.2 to 0.4 the similarity is low, from 0.4 to 0.6 the similarity is medium, from 0.6 to 0.8 high, and from 0.8 to 1 very high. We see that fuzzy similarity measure is medium (almost high).

5.4 Natural disasters, political stability, and political risk

We next consider the natural disaster risk, [61], the political stability, [87], and the political risk, [88], of OECD countries. We provide the rankings as given in [61,87,88].

The report in [61] systematically considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. The index is an average of several other indexes from the Economist Intelligence Unit, the Economic Forum, and the Political Risk Services, among others, [87]. The Political Risk Index is the overall measure of risk for a given country, calculated by using all 17 risk components from the PRS Methodology including turmoil, financial transfer, direct investment, and export markets. The Index provides a basic convenient way to compare countries directly as well as demonstrating changes over the last five years, [88].

The rankings in Table 5.2 are from low to high.

Let A denote the ranking of the countries with respect to natural disaster and let B denote the ranking of countries with respect to political stability. Let X denote the set of countries. Then $n = 36$. We have that

$$\begin{aligned} S_L &= S + \frac{1}{n}(S-1) \\ &= (\frac{n+1}{n})S - \frac{1}{n} \\ &= (\frac{n+1}{n})(1 - \frac{\sum_{x \epsilon X} |\mu_A(x) - \mu_B(x)|}{n(n+1)}) - \frac{1}{n} \\ &= 1 - \frac{\sum_{x \epsilon X} |\mu_A(x) - \mu_B(x)|}{n^2} \end{aligned}$$

Hence

$S_L(\mu_A, \mu_B) = 1 - \frac{1}{36^2}(149 + 141) = 1 - \frac{1}{1296}(290) = 0.7762$. The smallest $S_L(\mu_A, \mu_B)$ can be is 0.5. Thus $\frac{0.776-0.5}{1-0.5} = \frac{0.276}{0.5} = 0.552$. Hence the fuzzy similarity measure is medium.

We now consider Political Risk. Delete the countries that do not appear in all the rankings and rerank. Let A and B be the rankings of Natural Disaster and Political Stability, respectively, and C denote the ranking of Political Risk. Let X denote the set of these countries. Then $n = 26$. We have that $S_L(\mu_A, \mu_C) = 1 - \frac{208}{26^2} = 1 - \frac{208}{676} = 1 - 0.308 = 0.692$. The smallest S_L can be is $\frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2} + \frac{2}{0.352} = 0.500$. Hence $\frac{0.692-0.500}{1-0.500} = 0.384$. Thus fuzzy similarity measure is low.

We developed a new method to determine a fuzzy similarity method using fuzzy implication operators. We used this method to determine the fuzzy similarity between the two rankings of countries involving health security and health care. We then found a fuzzy similarity involving the rankings of countries with respect to national disaster and political disaster. In each case, we found the similarities to be medium. Future research could involve other regions in the world other than the OECD countries. Further reading on implication operators can be found in [29].

Table 5.2 Disaster stability risk.

Country	Natural Disaster	Political Stability	Political Risk
Australia	34	17	7
Austria	7	14	3.5
Belgium	19	24	20
Canada	33	12	1.5
Chile	30	32	18.5
Czech Rep.	3	9	3.5
Denmark	5	10	15.5
Estonia	11	19	
Finland	8	8	12
France	24	30	
Germany	17	20	15.5
Greece	25	31	
Hungary	2	15	23
Iceland	10	2	
Ireland	15	16	7
Israel	21	35	
Italy	26	25	25
Japan	32	6	10
Korea Rep.	28	23	18.5
Latvia	13	22	
Lithuania	14	18	
Luxembourg	1	3	
Mexico	36	34	
Netherlands	18	13	9
New Zealand	29	1	5
Norway	16	5	1.5
Poland	20	29	15.5
Portugal	22	11	24
Slovak Rep.	4	27	13
Slovenia	9	21	
Spain	27	26	21
Sweden	12	7	7
Switzerland	6	4	11
Turkey	31	36	26
United Kingdom	23	28	22
United States	35	33	15.5

5.5 Exercises

1. Find $S_L(\mu_B, \mu_C)$.

2. Find fuzzy similarity measures for other regions as in Table 5.1 for OECD countries.

CHAPTER

6 Fuzzy implication operators applied to country health[★]

Much of the work in this chapter is based on [78]. The Global Health Security Index states that all countries remain dangerously unprepared for future epidemic and pandemic threats, including threats potentially more devastating than COVID-19, [39]. In [76], we ranked the Organization for Economic Cooperation and Development (OECD) countries with respect to their preparation. In [25], countries are ranked with respect to their health care. We find the fuzzy similarity measure between these two rankings. We use implication operators to define a new fuzzy similarity measure to find the fuzzy similarity of these rankings. We also consider the natural disaster risk, the political stability of OECD countries. We provide the rankings as given in [61,87,88]. The report in [61] considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. We used five different fuzzy similarity measures. In three cases, we found the similarities to be medium and in two, we found the similarity to be low.

Define $\overline{\wedge} : [0,1] \times [0,1] \to [0,1]$ by $\overline{\wedge}(a,b) = 1$ if $a = b$ and $a \wedge b$ if $a \neq b$. Define $\varnothing : [0,1] \times [0,1] \to [0,1]$ by $\varnothing(a,b) = 1$ if $a = b$ and $\frac{a \wedge b}{a \vee b}$ if $a \neq b$. Note that for all $a, b \in [0,1]$, $\varnothing(a,b) = \frac{a \wedge b}{a \vee b}$.

6.1 Preliminary results

For ease of reading, we recall some definitions from the previous chapter. Let μ, ν be fuzzy subsets of a set X. Then M and S are fuzzy similarity measures on $\mathcal{FP}(X)$,

[★] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00014-3

where

$$M(\mu, \nu) = \frac{\sum_{x \in X} \mu(x) \wedge \nu(x)}{\sum_{x \in X} \mu(x) \vee \nu(x)},$$
$$S(\mu, \nu) = 1 - \frac{\sum_{x \in X} |\mu(x) - \nu(x)|}{\sum_{x \in X} (\mu(x) + \nu(x))}.$$

Results concerning fuzzy similarity measures can be found in [72,122].

Definition 6.1.1. [9, p. 14] Let I be a function of $[0, 1] \times [0, 1]$ into $[0, 1]$ such that $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$. Then I is called an **implication operator**.

An implication operator I is said to satisfy the **identity principle** if $I(x, x) = 1$ for all $x \in [0, 1]$. An implication operator is said to satisfy the **ordering principle** if $x \leq y \Leftrightarrow I(x, y) = 1$, [11]. Clearly, the ordering principle implies the identity principle.

I_1, I_2, and L defined below are implication operators that satisfy the ordering principle.

Example 6.1.2. Let $x, y \in [0, 1]$.
(1) Godel implication operator: $I_1(x, y) = 1$ if $x \leq y$, $I_1(x, y) = y$ otherwise.

(2) Goguen implication operator: $I_2(x, y) = 1$ if $x \leq y$ and $I_2(x, y) = y/x$ otherwise

(3) Luckasiewicz implication operator: $L(x, y) = (1 - x + y) \wedge 1$.

By [76, Theorem 3.1] S_L is a fuzzy similarity, where $S_L(\mu_A, \mu_B) = \frac{1}{n} \sum_{x \in X} (1 - \mu_A(x) \wedge (1 - \mu_B(x) \wedge (\mu_A(x) \wedge \mu_B(x))$.

Definition 6.1.3. [9, p. 15] Let I be an implication operator. Define the fuzzy subset E_I of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by for all $\mu, \nu \in \mathcal{FP}(X)$,

$$E_I(\mu, \nu) = \wedge\{\wedge\{I(\mu(x), \nu(x)) | x \in X\}, \wedge\{I(\nu(x), \mu(x)) | x \in X\}\}.$$

Then $E_I(\mu, \nu)$ is called the **degree of sameness** of μ and ν.

In [76], it was decided that the following definition would be more suitable than the previous definition for defining fuzzy similarity measures from implication operators.

Definition 6.1.4. Let I be an implication operator. Define $S : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0, 1]$ by for all $(\mu, \nu) \in \mathcal{FP}(X) \times \mathcal{FP}(X)$, $S(\mu, \nu) = \frac{1}{n} \sum_{x \in X} I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x)))$. Then S is called a **degree of likeness**.

In [76, Theorem 2.7], it was shown that the function S of Definition 6.1.4 is a fuzzy similarity measure.

An implication operator I is called a **hybrid monotonous implication operator** if $I(x, \_)$ is nondecreasing for all $x \in [0, 1]$ and $i(\_, y)$ is nonincreasing for all $y \in [0, 1]$.

Other implication operators can be found in [9].

Let X be a set with n elements, $n > 1$, say $X = \{x_1, ..., x_n)$. Let A be one-to-one function of X onto $\{1, ..., n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X by for all $x \in X$, $\mu_A(x) = \frac{A(x)}{n}$. Then μ_A is called the **fuzzy subset** associated with A. For two rankings A and B of X, $\sum_{x\in X}(A(x) + B(x)) = n(n+1)$ and so $\sum_{x\in X}(\mu_A(x) + \mu_B(x)) = n + 1$. Thus for S of Example 6.1.2,

$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x\in X} |\mu_A(x) - \nu_B(x)|}{n+1}.$$

6.2 Main results

Let S_1 and S_2 be the fuzzy similarity measures defined by I_1 and I_2 under Definition 6.1.4, respectively. Then

$$S_1(\mu, \nu) = \frac{1}{n}\sum_{x\in X} \mu(x) \mathbin{\overline{\wedge}} \nu(x),$$
$$S_2(\mu, \nu) = \frac{1}{n}\sum_{x\in X} \mu(x) \varnothing \nu(x).$$

We next consider how small S_1 can be with respect to rankings A and B.

Suppose n is even. Let A be the ranking: $1, 2, ..., \frac{n}{2}, \frac{n+2}{2}, ..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, ..., 2, 1$. Then

$$S_1(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\in X} \mu_A(x) \mathbin{\overline{\wedge}} \mu_B(x) = \frac{1}{n}(2(1+2+...+\frac{n}{2}))\frac{1}{n}$$
$$= \frac{1}{n}(2([\frac{n}{2}(\frac{n}{2}+1)])/2)\frac{1}{n} = \frac{1}{n^2}(\frac{n^2}{4} + \frac{n}{2}) = \frac{1}{4} + \frac{1}{2n}.$$

Suppose that n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then

$$S_1(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\in X} \mu_A(x) \mathbin{\overline{\wedge}} \mu_B(x) = \frac{1}{n}(1 + 2(1+2+...+\frac{n-1}{2}))\frac{1}{n}$$
$$= \frac{1}{n^2}(1 + 2(\frac{n-1}{2})(\frac{n-1}{2}+1)/2) = \frac{1}{n^2}(1 + 2(\frac{n-1}{2}\frac{n+1}{2}\frac{1}{2}))$$
$$= \frac{1}{n^2}(1 + \frac{n^2-1}{4}) = \frac{1}{n^2} + \frac{1}{4} - \frac{1}{4n^2} = \frac{1}{4} + \frac{3}{4n^2}.$$

Example 6.2.1. Let $n = 6$. Let A be the ranking $1, 2, ..., 5, 6$ and B the ranking $6, ..., 2, 1$. Then $\mu_A(x_i) = \frac{i}{6}$ and $B(x_i) = \frac{6-i+1}{6}$, $i = 1, 2, ..., 6$. Hence

$$\frac{\mu_A(x_1) \wedge \mu_B(x_1)}{\mu_A(x_1) \vee \mu_B(x_1)} = \frac{\frac{1}{6}}{\frac{6}{6}} = \frac{\mu_A(x_6) \wedge \mu_B(x_6)}{\mu_A(x_6) \vee \mu_B(x_6)},$$

$$\frac{\mu_A(x_2) \wedge \mu_B(x_2)}{\mu_A(x_2) \vee \mu_B(x_2)} = \frac{\frac{2}{6}}{\frac{5}{6}} = \frac{\mu_A(x_5) \wedge \mu_B(x_5)}{\mu_A(x_5) \vee \mu_B(x_5)},$$

$$\frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)} = \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{\mu_A(x_4) \wedge \mu_B(x_4)}{\mu_A(x_4) \vee \mu_B(x_4)}.$$

Let $n = 5$. Let A be the ranking $1, 2, ..., 5$, and B the ranking $5, ..., 2, 1$. Then $\mu_A(x_i) = \frac{i}{5}$ and $B(x_i) = \frac{5-i+1}{5}$, $i = 1, 2, ..., 5$. Hence

$$\frac{\mu_A(x_1) \wedge \mu_B(x_1)}{\mu_A(x_1) \vee \mu_B(x_1)} = \frac{\frac{1}{5}}{\frac{5}{5}} = \frac{\mu_A(x_5) \wedge \mu_B(x_5)}{\mu_A(x_5) \vee \mu_B(x_5)},$$

$$\frac{\mu_A(x_2) \wedge \mu_B(x_2)}{\mu_A(x_2) \vee \mu_B(x_2)} = \frac{\frac{2}{5}}{\frac{4}{5}} = \frac{\mu_A(x_4) \wedge \mu_B(x_4)}{\mu_A(x_4) \vee \mu_B(x_4)},$$

$$\frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)} = \frac{\frac{3}{5}}{\frac{3}{5}} = \frac{\mu_A(x_3) \wedge \mu_B(x_3)}{\mu_A(x_3) \vee \mu_B(x_3)}.$$

We see that for n odd, the middle term will yield the value 1.

The following discussion is to determine the smallest value a fuzzy similarity measure can be with respect to rankings. Let S be any fuzzy similarity measure with respect to some rankings A and B. We determine the smallest value S can be for the following reason: Say, the smallest value S can be is S^*. Then the ratio $\frac{S-S^*}{1-S^*}$ ranges from 0 to 1. A clearer picture of the similarity is thus provided.

Lemma 6.2.2. (1) *Suppose n is even. Let A be the ranking:* $1, 2, ..., \frac{n}{2}, \frac{n+2}{2}, ..., n-1, n$ *and let B be the ranking* $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, ..., 2, 1$. *Then* $\frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n}) = (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}$.

(2) *Suppose n is odd. Let A be the ranking* $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ *and B be the ranking* $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. *Then* $\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n} = (n+1)\sum_{j=\frac{n+1}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}$.

Proof. (1) $\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n} =$

$$\sum_{i=1}^{\frac{n}{2}} \frac{i}{n-i+1} = \sum_{j=\frac{n}{2}+1}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n}{2}+1}^{n} (\frac{n}{j} - 1 + \frac{1}{j})$$

$$= (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}.$$

(2) $\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + \ldots + \frac{2}{n-1} + \frac{1}{n} = \sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1}$.

Let $j = n - i + 1$. Then $i = n - j + 1$ and $j = n, n-1, \ldots, \frac{n}{2} + \frac{3}{2}$. Now

$$\sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1} = \sum_{j=\frac{n+3}{2}}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n+3}{2}}^{n} (\frac{n}{j} - 1 + \frac{1}{j})$$

$$= (n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}.$$ ∎

Theorem 6.2.3. (1) *Suppose n is even. Let A be the ranking: $1, 2, \ldots \frac{n}{2}, \frac{n+2}{2}, \ldots, n-1, n$ and let B be the ranking $n, n-1, \ldots, \frac{n+2}{2}, \frac{n}{2}, \ldots, 2, 1$. Then $S_2(\mu_A, \mu_B) = \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}]$.*

(2) *Suppose n is odd. Let A be the ranking $1, 2, \ldots, \frac{n+1}{2}, \ldots, n-1, n$ and B be the ranking $n, n-1, \ldots, \frac{n+1}{2}, \ldots, 2, 1$. Then $S_2(\mu_A, \mu_B) = \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1]$.*

Proof. (1) $S_2(\mu_A, \mu_B) = \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n+2}{2}} + \ldots + \frac{2}{n-1} + \frac{1}{n})2 = \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + \ldots + \frac{2}{n-1} + \frac{1}{n})2 = \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}]$ by Lemma 6.2.2 (1).

(2) $S_2(\mu_A, \mu_B) = \frac{1}{n}((\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + \ldots + \frac{2}{n-1} + \frac{1}{n})2 + 1) = \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1]$ by Lemma 6.2.2 (2). ∎

We next determine approximate values for $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$ when n is even and $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$ when n is odd. Recall that $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is a harmonic sum which sums approximately to $\gamma + \ln 2$, where γ is the Euler-Mascheroni constant, $\gamma \approx 0.5772$ and where $\approx$ denotes approximately equal to.

Let n be even. Consider $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n}{2}) = \ln n - \ln \frac{n}{2} = \ln 2$.

Let n be odd. Consider $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n+1}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n+1}{2}) = \ln n - \ln \frac{n+1}{2} = \ln \frac{2n}{n+1}$.

Theorem 6.2.4. (1) *Suppose n is even. Let A be the ranking:* $1, 2, ...\frac{n}{2}, \frac{n+2}{2}..., n-1, n$ *and let B be the ranking* $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, ..., 2, 1$. *Then* $S_2(\mu_A, \mu_B) \approx 0.386 + \frac{2}{n}\ln 2$.

(2) *Suppose n is odd. Let A be the ranking* $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ *and B be the ranking* $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. *Then* $S_2(\mu_A, \mu_B) \approx 2\ln\frac{2n}{n+1} + \frac{2}{n}\ln\frac{2n}{n+1} - 1 + \frac{2}{n}$.

Proof. Theorem 6.2.3 is used in the following arguments.

(1) We have

$$
\begin{aligned}
S_2(\mu_A, \mu_B) &= \frac{1}{n}(\sum_{j=1}^{\frac{n}{2}} \frac{j}{n-j+1})2 \\
&= \frac{2}{n}[(n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}] \\
&\approx \frac{2}{n}[(n+1)\ln 2 - \frac{n}{2}] \\
&= (2 + \frac{2}{n})\ln 2 - 1 \\
&= 2\ln 2 + \frac{2}{n}\ln 2 - 1 \\
&\approx 0.386 + \frac{2}{n}\ln 2.
\end{aligned}
$$

(2) We have

$$
\begin{aligned}
S_2(\mu_A, \mu_B) &= \frac{1}{n}[[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}]2 + 1] \\
&= \frac{2}{n}[(n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}] + \frac{1}{n} \\
&\approx \frac{2}{n}[(n+1)\ln\frac{2n}{n+1} - \frac{n-1}{2}] + \frac{1}{n} \\
&= (2 + \frac{2}{n})\ln\frac{2n}{n+1} - (1 - \frac{1}{n}) + \frac{1}{n} \\
&= 2\ln\frac{2n}{n+1} + \frac{2}{n}\ln\frac{2n}{n+1} - 1 + \frac{2}{n}.
\end{aligned}
$$

■

Proposition 6.2.5. *Let* $S_1, ..., S_n$ *be fuzzy similarity measures on* $\mathcal{FP}(X)$. *Let* $w_i \in [0, 1]$ *be such that* $\sum_{i=1}^{n} w_i = 1$. *Then* $\sum_{i=1}^{n} w_i S_i$ *is a fuzzy similarity measure on* $\mathcal{FP}(X)$.

Proof. Let $S = \sum_{i=1}^{n} w_i S_i$ and $\mu, \nu, \rho \in \mathcal{FP}(X)$. Then $S(\mu, \nu) = \sum_{i=1}^{n} w_i S_i(\mu, \nu) = \sum_{i=1}^{n} w_i S_i(\nu, \mu) = S(\nu, \mu)$. Now $S(\mu, \nu) = 1 \Leftrightarrow \sum_{i=1}^{n} w_i S_i(\mu, \nu) = 1 \Leftrightarrow S_i(\mu, \nu) = 1$ for $i = 1, \ldots, n \Leftrightarrow \mu = \nu$. Suppose that $\mu \subseteq \nu \subseteq \rho$. Then $S_i(\mu, \rho) \leq S_i(\mu, \nu) \wedge S_i(\nu, \rho)$, $i = 1, \ldots, n$. Hence

$$\sum_{i=1}^{n} w_i S_i(\mu, \rho) \leq \sum_{i=1}^{n} w_i [S_i(\mu, \nu) \wedge S_i(\nu, \rho)] = \sum_{i=1}^{n} w_i S_i(\mu, \nu) \wedge w_i S_i(\nu, \rho)$$
$$\leq \sum_{i=1}^{n} w_i S_i(\mu, \nu) \wedge \sum_{i=1}^{n} w_i S_i(\nu, \rho) = S(\mu, \nu) \wedge S(\nu, \rho).$$

Suppose $S(\mu, \nu) = 0$. Then $\sum_{i=1}^{n} w_i S_i(\mu, \nu) = 0$. Thus $S_i(\mu, \nu) = 0$ for all i such that $w_i > 0$. Thus for all $x \in X$, $\mu(x) \wedge \nu(x) = 0$. ■

Proposition 6.2.6. *Let $S_1, \ldots, S_n$ be fuzzy similarity measures on $\mathcal{FP}(X)$. Let $w_i \in [0, 1]$ be such that $\sum_{i=1}^{n} w_i = 1$. Let a_i be the smallest value S_i can be, $i = 1, \ldots, n$. Then $\sum_{i=1}^{n} w_i a_i$ is the smallest value $\sum_{i=1}^{n} w_i S_i$ can be.*

Proof. Suppose $(\sum_{i=1}^{n} w_i S_i)(\mu, \nu) = b$. Then $\sum_{i=1}^{n} (w_i S_i)(\mu, \nu) = b$. Let $S_i(\mu, \nu) = b_i$, $i = 1, \ldots, n$. Then $b_i \geq a_i$, $i = 1, \ldots, n$. Now $b = \sum_{i=1}^{n} w_i b_i$ and so $b \geq \sum_{i=1}^{n} w_i a_i$. ■

Converting a fuzzy similarity measures to a measure using the smallest value it can be, converts the measure to the interval [0, 1]. We can say if this converted value lies between 0 and 0.2, the similarity is very low, from 0.2 to 0.4 the similarity is low, from 0.4 to 0.6 the similarity is medium, from 0.6 to 0.8 high, and from 0.8 to 1 very high.

6.3 Country health

The 2021 Global Health Security Index measures the capacities of 195 countries to prepare for epidemics and pandemics. All countries remain dangerously unprepared for future epidemics and pandemic threats, including threats potentially more devastating than COVID-19, [25]. In [39], a ranking of countries with respect to health care is provided. We provide the ranking with respect to OECD countries (Table 6.1).

Let M and S be the fuzzy similarity measures defined in Section 6.1. We deleted the countries in the Health Security ranking that were not in the Health Care ranking and then reranked the Health Security countries. We found that $S(\mu_A, \mu_B) = 1 - \frac{223}{1122} = 1 - 0.199 = 0.801$. By [73, Theorem 2.10], $S(\mu_A, \mu_B) = \frac{2M(\mu_A, \mu_B)}{1+M(\mu_A, \mu_B)}$. Hence $M(\mu_A, \mu_B) = \frac{S(\mu_A, \mu_B)}{2-S(\mu_A, \mu_B)} = \frac{0.801}{1.199} = 0.668$. With the countries deleted, $n = 33$. Thus the smallest M can be is $\frac{n+1}{3n-1} = \frac{34}{98} = 0.347$. The smallest S can be is $\frac{1}{2} + \frac{1}{2n} =$

Table 6.1 OECD health security and health care rankings.

Country	Health Security	Health Care	Country	Health Security	Health Care
Australia	2	9	Korea, Rep.	8	1
Austria	22	5	Latvia	14	
Belgium	19	4	Lithuania	18	26
Canada	4	19	Luxembourg	35	
Chile	23	30	Mexico	21	23
Czech Rep.	30	12	Netherlands	10	3
Denmark	11	8	New Zealand	12	16
Estonia	25	18	Norway	17	13
Finland	3	11	Poland	24	29
France	13	6	Portugal	27	22
Germany	7	10	Slovak Rep.	29	28
Greece	32	31	Slovenia	5	27
Hungary	28	33	Spain	15	7
Iceland	34		Sweden	9	20
Ireland	26	32	Switzerland	20	17
Israel	36	15	Turkey	33	21
Italy	31	25	United Kingdom	6	14
Japan	16	2	United States	1	24

$\frac{1}{2} + \frac{1}{66} = 0.515$. Therefore

$$\frac{0.668 - 0.347}{1 - 0.347} = \frac{0.321}{0.653} = 0.492$$

and

$$\frac{0.801 - 0.515}{1 - 0.515} = \frac{0.286}{0.485} = 0.590.$$

We see that in both cases the similarity is medium.

A fuzzy similarity measure using implication operators was defined in [76]: $S_L(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\in X}[(1-\mu_A(x)) \wedge (1-\mu_B(x)) + \mu_A(x) \wedge \mu_B(x)]$. We have by [2, Proposition 3.5] that $S_L = S + \frac{1}{n}(S-1)$. Thus $S_L(\mu_A, \mu_B) = 0.801 + \frac{1}{33}(0.801 - 1) = 0.801 - 0.006 = 0.795$. The smallest $S_L(\mu_A, \mu_B)$ is $\frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2} + \frac{1}{2178} = 0.5 + 0.000459$ which we round off to 0.5. Thus $\frac{0.795-0.5}{1-0.5} = 0.59$. The similarity is thus medium.

We have that

$$\begin{aligned} S_1(\mu_A, \mu_B) &= \frac{1}{n}\sum_{x\in X} \mu_A \overline{\wedge} \mu_B(x) \\ &= \frac{1}{33}(\frac{428}{33}) = 0.393. \end{aligned}$$

The smallest S_1 can be is $\approx \frac{1}{4} + \frac{3}{4n^2} = 0.25 + 0.003 = 0.253$. Thus $\frac{0.393-0.253}{1-0.253} = \frac{0.140}{0.747} = 0.187$ and so the similarity is very low.

We find that $S_2(\mu_A, \mu_B) = \frac{17.921}{33} = 0.543$. The smallest S_2 can be is $\approx 2\ln\frac{66}{34} + \frac{2}{34} - 1 + \frac{3}{33} = 0.426$. Thus we have $\frac{0.543-0.426}{1-0.426} = \frac{0.117}{0.514} = 0.228$.

Hence the similarity is low.

We have that $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L \approx \frac{1}{3}(0.393 + 0.543 + 0.795) = \frac{1}{3}(1.649) = 0.577$. The smallest $\frac{1}{3}S_1 + \frac{1}{3}S_2 + \frac{1}{3}S_L$ can be is $\approx \frac{1}{3}(0.276 + 0.426 + 0.500) = \frac{1}{3}(1.202) = 0.401$.

Now $\frac{0.577-0.401}{1-0.401} = \frac{0.176}{0.499} = 0.353$. Here the similarity is low.

6.4 Natural disaster, political stability, and political risk

We next consider the natural disaster risk, [61], the political stability, [87], and the political risk, [88], of OECD countries. We provide the rankings as given in [61,87,88]. The report in [61] systematically considers a country's vulnerability and exposure to natural hazards to determine a ranking of countries around the world based on their natural disaster risk. The index of Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood that the government will be destabilized or overthrown by unconstitutional or violent means, including politically motivated violence and terrorism. The index is an average of several other indexes from the Economist Intelligence Unit, the Economic Forum, and the Political Risk Services, among others, [87]. The Political Risk Index is the overall measure of risk for a given country, calculated by using all 17 risk components from the PRS Methodology, including turmoil, financial transfer, direct investment, and export markets. The Index provides a basic convenient way to compare countries directly as well as demonstrating changes over the last five years, [88].

The rankings in Table 6.2 are from low to high.

Let M and S be the fuzzy similarity measures defined in Section 6.1. Here $n = 36$. We have that $S(\mu_A, \mu_B) = 1 - \frac{290}{1332} = 0.782$. Thus $M(\mu_A, \mu_B) = \frac{S(\mu_A,\mu_B)}{2-S(\mu_A,\mu_B)} = \frac{0.782}{1.218} = 0.642$. The smallest M can be is $\frac{n+2}{3n+2} = \frac{38}{110} = 0.345$. Hence $\frac{0.642-0.345}{1-0.345} = 0.453$. Therefore, the similarity is medium. The smallest S can be is $\frac{n/2+1}{n+1} = \frac{19}{37} = 0.514$. Thus $\frac{0.782-0.514}{1-0.514} = 0.551$. Hence the similarity is medium.

$S_L(\mu_A, \mu_B) = 1 - \frac{1}{36^2}(149 + 141) = 1 - \frac{1}{1296}(290) = 0.7762$. The smallest $S_L(\mu_A, \mu_B)$ can be is 0.5. Thus $\frac{0.776-0.5}{1-0.5} = \frac{0.276}{0.5} = 0.552$. Hence the fuzzy similarity measure is medium.

$S_1(\mu_A, \mu_B) = \frac{\sum_{x\in X}\mu_A(x)\overline{\wedge}\mu_B(x)}{n} = \frac{549/36}{36} = 0.424$ and $S_2 \approx S_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} = 0.424 + 0.068 = 0.492$.

Table 6.2 OECD natural disaster and political stability rankings.

Country	Natural Disaster	Political Stability	Country	Natural Disaster	Political Stability
Australia	34	17	Korea, Rep.	28	23
Austria	7	14	Latvia	13	22
Belgium	19	24	Lithuania	14	18
Canada	33	12	Luxembourg	1	3
Chile	30	32	Mexico	36	34
Czech Rep.	3	9	Netherlands	18	13
Denmark	5	10	New Zealand	29	1
Estonia	11	19	Norway	16	5
Finland	8	8	Poland	20	29
France	24	30	Portugal	22	11
Germany	17	20	Slovak Rep.	4	27
Greece	25	31	Slovenia	9	21
Hungary	2	15	Spain	27	26
Iceland	10	2	Sweden	12	7
Ireland	15	16	Switzerland	6	4
Israel	21	35	Turkey	31	36
Italy	26	25	United Kingdom	23	28
Japan	32	6	United States	35	33

The smallest S_1 can be is $\frac{1}{4}+\frac{1}{n}=0.25+0.028=0.278$ since $n=36$ is even. Thus $\frac{0.492-0.278}{1-0.278}=\frac{0.214}{0.722}=0.296$. Hence the similarity is low.

We find that $S_2(\mu_A,\mu_B)=\frac{22}{36}=0.611$. The smallest S_2 can be is $\approx 0.386+\frac{2}{36}(0.693)=0.442$. Thus $\frac{0.611-0.442}{1-0.442}=\frac{0.169}{0.558}=0.303$. Once again the similarity is low.

We have that $\frac{1}{3}S_1+\frac{1}{3}S_2+\frac{1}{3}S_L\approx\frac{1}{3}(0.424+0.611+0.776)=\frac{1}{3}(1.811)=0.604$. The smallest $\frac{1}{3}S_1+\frac{1}{3}S_2+\frac{1}{3}S_L$ can be is $\approx\frac{1}{3}(0.278+0.442+0.500)=\frac{1}{3}(1.220)=0.407$.

Now $\frac{0.604-0.407}{1-0.402}=\frac{0.197}{0.598}=0.329$. The average similarity is low.

We used fuzzy implication operators to define the fuzzy similarity between the two rankings of countries involving health security and health care. We then found a fuzzy similarity involving the rankings of countries with respect to national disaster and political disaster. In each case, we found the similarity measures to be medium for S_L, M, and S and low for S_1 and S_2. Future research could involve other regions in the world other than the OECD countries. It was shown in [39, Theorem 3.6], that $M\subseteq S_L\subseteq S$. It is clear that $S_1\subseteq S_2$. Another potential project is to determine the relationship between S_2 and M. Further reading on implication operators can be found in [9].

6.5 Sustainable development goals and air pollution

The World Health Organization (WHO) works to ensure that health-relevant indicators of household and ambient pollution exposure and burden of disease are included in the formal system SDG indicators. WHO monitors and tracks progress of health indicators to measure progress toward achieving the SDG3 on health, SDG7 on energy, and SDG11 on cities. In Table 6.3 we rank how well countries are achieving SDG 3, 7, 11.

In [22], how tackling air pollution contributes to achieving the SDGs. It is stated in [22] that clean air can be a catalyst for driving progress on agenda 2030. We won't be able to meet those goals and ensure no one is left behind if air pollution is not addressed. The SDGs that can be accelerated by clean air are listed in [22], SDG 3 (Good Health and Well-Being), SDG 7 (Affordable and Clean Energy), SDG 2 (Zero Hunger), SDG 4 (Quality Education), SDG 5 (Gender Equality), SDG 8 (Decent Work and Economic Growth), SDG 11 (Sustainable Cities and Communities), SDG 12 (Responsible Construction and Production), SDG 13 (Climate Action) and SDG 15 (Life on Land).

Air pollution can cause respiratory and cardiovascular diseases, which are major contributors to global mortality rates.

In the column 4 of Table 6.3, we present the region of which the country is a member.

OECD Organization for Economic Cooperation and Development

ESA East and South Asia

EECA Eastern Europe and Central Asia

LAC Latin America and the Caribbean

MENA Middle East and North Africa.

SSA Sub-Saharan Africa

In column 3, we present the average of the achievement values of the countries for SDG 3, SDG 7, and SDG 11 as determined in [71]. In column 4, we place the countries in their region and then provide the rank of the countries for their particular region.

In the above table, we ranked the OECD countries with respect to their achievement of SDGs 3, 7, 11. We also reversed the rankings for the OECD countries so that a small number represented low pollution. Let A denote this ranking and Let B denote the SDG 3, 7, and 11 ranking. We next determine the fuzzy similarity measures

Table 6.3 Air pollution.

Country	Most Polluted	SDG 3, 7, 11	Region
Chad	1	0.205	SSA
Iraq	2	0.700	MENA
Pakistan	3	0.577	ESA
Bahrain	4	0.787	MENA
Bangladesh	5	0.558	ESA
Burkino Faso	6	0.383	SSA
Kuwait	7	0.749	MENA
India	8	0.584	ESA
Egypt	9	0.738	MENA
Tajikistan	10	0.805	EECA
United Arab Emirates	11	0.843	MENA
Sudan	12	0.480	SSA
Rwanda	13	0.432	SSA
Qatar	14	0.705	MENA
Saudi Arabia	15	0.705	MENA
Nepal	16	0.556	ESA
Uganda	17	0.343	SSA
Nigeria	18	0.326	SSA
Bosnia Herzegovina	19	0.775	EECA
Uzbekistan	20	0.855	EECA
Iran	21	0.804	MENA
Armenia	22	0.804	EECA
Ethiopia	23	0.486	SSA
Kyrgyzstan	24	0.824	EECA
China	25	0.777	ESA
Indonesia	26	0.679	ESA
Ghana	27	0.543	SSA
Mongolia	28	0.564	ESA
Laos	29	0.734	ESA
Vietnam	30	0.783	ESA
North Macedonia	31	0.834	EECA
Gabon	32	0.617	SSA
Serbia	33	0.803	EECA
Zambia	34	0.566	SSA
Myanmar	35	0.571	ESA
Madagascar	36	0.371	SSA
Croatia	37	0.848	EECA
Peru	38	0.783	LAC
South Africa	39	0.685	SSA

continued on next page

Table 6.3 (*continued*)

Country	Most Polluted	SDG 3, 7, 11	Region
Kazakhstan	40	0.800	EECA
Moldova	41	0.815	EECA
Ivory Coast	42		
Chile	43	0.861	OECD /30
Turkmenistan	44	0.739	EECA
Turkey	45	0.811	OECD/36
Sri Lanka	46	0.749	ESA
Senegal	47	0.552	SSA
Syria	48	0.675	MENA
Mexico	49	0.832	OECD/35
Greece	50	0.877	OECD/26
Azerbaijan	51	0.831	EECA
Italy	52	0.874	OECD/28
Israel	53	0.834	OECD/34
Guatemala	54	0.752	LAC
Bulgaria	55	0.841	EECA
South Korea	56	0.884	OECD/22.5
Thailand	57	0.811	ESA
Algeria	58	0.760	MENA
Malaysia	59	0.839	ESA
Romania	60	0.836	EECA
Georgia	61	0.830	EECA
Poland	62	0.853	OECD /32
Columbia	63	0.840	LAC
Montenegro	64	0.777	EECA
Cyprus	65	0.876	EECA
Dem. Rep. of the Congo	66		
Macao SAR	67		
Slovenia	68	0.907	OECD/15.5
Philippines	69	0.671	ESA
Kosovo	70		
Slovakia	71	0.874	OECD/27
Hong Kong (SAR)	72		
Albania	73	0.831	SSCA
El Salvador	74	0.849	LAC
Czech Rep.	75	0.912	OECD/13.5
Taiwan	76		
Singapore	77	0.948	ESA
Lithuania	78	0.837	OECD/33

continued on next page

Table 6.3 (*continued*)

Country	Most Polluted	SDG 3, 7, 11	Region
Guyana	79	0.743	LAC
Hungary	80	0.879	OECD/24.5
Brazil	81	0.831	LAC
Malta	82	0.894	EECA
Kenya	83	0.529	SSA
France	84	0.928	OECD/11
Uruguay	85	0.881	LAC
Russia	86	0.839	EECA
Netherlands	87	0.930	OECD/9.5
Germany	88	0.930	OECD/9.5
Spain	89	0.931	OECD/8
Maldives	90	0.947	ESA
Belgium	91	0.894	OECD/20
Austria	92	0.915	OECD/12
Honduras	93	0.765	LAC
Latvia	94	0.873	OECD/29
Switzerland	95	0.976	OECD/1
Ukraine	96	0.798	EECA
Japan	97	0.879	OECD/24.5
Panama	98	0.851	LAC
USA	99	0.884	OECD/22.5
Nicaragua	100	0.750	LAC
United Kingdom	101	0.952	OECD/4
Angola	102	0.428	SSA
Denmark	103	0.933	OECD/7
Cambodia	104	0.611	ESA
Liechtenstein	105		
Portugal	106	0.904	OECD/17
Costa Rica	107	0.900	LAC
Argentina	108	0.850	LAC
Ireland	109	0.907	OECD/15.5
Luxembourg	110	0.859	OECD/31
Canada	111	0.902	OECD/18
Bolivia	112	0.740	LAC
Suriname	113	0.787	LAC
Norway	114	0.942	OECD/5
Sweden	115	0.956	OECD/2
Belize	116	0.775	LAC
Andorra	117		

continued on next page

Table 6.3 (*continued*)

Country	Most Polluted	SDG 3, 7, 11	Region
Trinidad and Tobago	118	0.769	LAC
Finland	119	0.936	OECD/6
Estonia	120	0.893	OECD/21
New Zealand	121	0.912	OECD/13.5
Puerto Rico	122		
Australia	123	0.897	OECD/19
Grenada	124		
New Caledonia	125		
Iceland	126	0.954	OECD/3
Bonaire, St. Eustatius and Saba	127		
Burundi	128	0.347	SSA
U.S. Virgin Islands	129		
French Polynesia	130		
Guam	131		

of A and B. Let X denote the set of OECD countries. Then $n = 36$. We have that

$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x \epsilon X} |\mu_A(x) - \mu_B(x)|}{\sum_{x \epsilon X} (\mu_A(x) + \mu_B(x))}$$
$$= 1 - \frac{277}{1332}$$
$$= 0.792.$$

The smallest S can be is $\frac{n/2+1}{n+1} = \frac{19}{37} = 0.51$. Thus $\frac{0.792-0.51}{1-0.51} = \frac{0.282}{0.49} = 0.576$.

Now $M = \frac{S}{2-S}$. Hence

$$M(\mu_A, \mu_B) = \frac{0.792}{2 - 0.792}$$
$$= 0.656.$$

The smallest M can be is $\frac{0.51}{2-0.51} = 0.342$. Thus $\frac{0.656-0.342}{1-0.342} = \frac{0.314}{0.658} = 0.477$.

We see that the fuzzy similarity measure is medium in both cases.

6.6 Exercises

1. Rank how well countries are achieving these SDGs, SDG 3, (Good Health and Well-Being), SDG 7 (Affordable and Clean Energy), SDG 2 (Zero Hunger), SDG 4 (Quality Education), SDG 5 (Gender Equality), SDG 8 (Decent Work and Economic

Growth), SDG 11 (Sustainable Cities and Communities), SDG 12 (Responsible Construction and Production), SDG 13 (Climate Action), SDG 15 (Life on Land). Proceed as in Table 6.3 with SDG 3, SDG 7, and SDG 11.

CHAPTER

Mistreatment of women and children[*]

7

In this chapter, we examine the mistreatment of women and children using fuzzy implication operators. We show how fuzzy implication operators can be used to define fuzzy similarity measures. These fuzzy similarity measures are used to compare the similarity of various rankings of countries with respect to the security status, gender equality, and human development of women and children. We prove relationships between certain fuzzy implication operators. The results in this chapter rely heavily on [81].

The following is taken from [30]. Technology-facilitated violence: Available evidence collected at country and regional levels confirms high prevalence rates against women and girls. One in 10 women in the European union has experienced cyber-harassment since the age of 15. In the Arab States, a regional study found that 60% of women internet users in the region had been exposed to online violence. In Uganda, in 2021, 49% of women reported being involved in online harassment at some point in their lifetime. According to a 2016 survey by the Korean National Human Rights Commission, 85% of women experienced hate speech online.

Climate change and violence: Climate change and slow environmental degradation exacerbate the risks of violence against women and girls due to displacement, resource scarcity and food insecurity and disruption to service provision for survivors. Following Hurricane Katrina in 2005, the rate of rape among women displaced to trailer parks rose 53.6 times the baseline rate in Mississippi, USA, for that rate. In Ethiopia, there was an increase in girls sold into early marriage in exchange for livestock to help families cope with the impacts of prolonged droughts. Nepal witnessed an increase in trafficking from an estimated 3,000–5,000 annually in 1990 to 12,000–20,000 per year after the 2015 earthquake.

Trafficking in women: In 2020, for every 10 victims of human trafficking detected globally, about four were women and about two were girls. Most of the detected victims of trafficking for sexual exploitation (91%) are women.

The study in [30] also considered femicides/feminicides, prevalence of violence against women and girls, impact of COVID-19 on violence against women and girls,

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00015-5

reporting violence against women, laws on violence against women and girls, and many other categories.

The following is from [36]. Gender-based violence occurs in every country in the world and across all economic and social groups. One in three women and girls will experience sexual or physical violence in their lifetimes. Gender-based violence has been ingrained into society, in some countries and regions more than others. In many communities, violence against girls and women is expected and even accepted. The military use of schools continues in Syria, Yemen, Sudan, the Philippines and Afghanistan. In some contexts, schoolgirls have been specifically targeted for sexual violence and by armed groups who oppose female education. Some global trends are 15 million girls are married before the age of 18, 30 million girls are at risk of female genital mutilation in the next decade, and 1 in 3 girls and women live in countries where marital rape is not an explicit crime. Due to their gender, girls are often forced to drop out of school, are prevented from accessing income-generating opportunities, and ultimately face social exclusion. More information can be found in [36]. See also [97,120].

The Women, Peace and Security (WPS) Index ranks 177 countries and economies on women's status, [114]. Countries are also ranked according to their achievement of the rights of a child, [90]. The Gender Inequality Index (GII) ranks countries with respect to the loss of achievement within a country due to gender inequality, [37]. The Human Development Index (HDI) ranks countries with respect to human development. In this chapter, we determine the similarity of the rankings using various fuzzy similarity measures.

7.1 Preliminary results

Let μ, ν be fuzzy subsets of a set X. Then M and S are fuzzy similarity measures on $\mathcal{FP}(X)$, where

$$M(\mu,\nu) = \frac{\sum_{x\in X}\mu(x)\wedge\nu(x)}{\sum_{x\in X}\mu(x)\vee\nu(x)},$$

$$S(\mu,\nu) = 1 - \frac{\sum_{x\in X}|\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))}.$$

Recall a function I of $[0,1]\times[0,1]$ into $[0,1]$ such that $I(0,0)=I(0,1)=I(1,1)=1$ and $I(1,0)=0$ is called an implication operator. Other definitions can be found in the previous chapter.

Let A be the ranking $1,2,\ldots,n$ and B be the ranking $n,\ldots,2,1$. For n even, we have $I_1(\mu_A,\mu_B)=\frac{1}{n}(1+\ldots+1+\frac{n}{2}+\ldots+2+1)\frac{1}{n}$ and $I_2(\mu_A,\mu_B)=\frac{1}{n}(1+\ldots+1+\frac{\frac{n}{2}}{\frac{n+2}{2}}+\ldots+\frac{2}{n-1}+\frac{1}{n})$. For n odd, we have $I_1(\mu_A,\mu_B)=\frac{1}{n}(1+\ldots+1+\frac{n+1}{2}+\ldots+2+1)\frac{1}{n}$ and $I_2(\mu_A,\mu_B)=\frac{1}{n}(1+\ldots+1+\frac{\frac{n-1}{2}}{\frac{n+3}{2}}+\ldots+\frac{2}{n-1}+\frac{1}{n})$.

We determine the smallest value a fuzzy similarity measure S can be with respect to rankings since then the ratio $(S\text{-min})/(\max - \min)$.

By [76, Theorem 3.1], S_L is a fuzzy similarity, where $S_L(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\in X}(1-\mu_A(x)) \wedge (1-\mu_B(x) \wedge (\mu_A(x) \wedge \mu_B(x)$.

Recall the following definition. Let I be an implication operator. Define the fuzzy subset E_I of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ by for all $\mu, \nu \in \mathcal{FP}(X)$,

$$E_I(\mu, \nu) = \wedge\{\wedge\{I(\mu(x), \nu(x))|x \in X\}, \wedge\{I(\nu(x), \mu(x))|x \in X\}\}.$$

Then $E_I(\mu, \nu)$ is called the **degree of sameness** of μ and ν.

Proposition 7.1.1. *Let I be a hybrid monotonous implication operator that satisfies the ordering principle. Then E_I satisfies the following properties $\forall \mu, \nu, \rho \in \mathcal{FP}(X)$:*

(1) *$E_I(\mu, \nu) = E_I(\nu, \mu)$;*

(2) *$S(\mu, \nu) = 1$ implies $\mu = \nu$;*

(3) *If $\mu \subseteq \nu \subseteq \rho$, then $S(\mu, \rho) \le S(\mu, \nu) \wedge S(\nu, \rho)$.*

Proof. (1) Clearly, $E_I(\mu, \nu) = E_I(\nu, \mu)$.

(2) $E_I(\mu, \nu) = 1 \Leftrightarrow \wedge\{I(\mu(x), \nu(x))|x \in X\} = 1$ and $\wedge\{I(\nu(x), \mu(x))|x \in X\}\} = 1 \Leftrightarrow \mu(x) = \nu(x) = 1 \forall x \in X \Rightarrow \mu = \nu$.

(3) Suppose $\mu \subseteq \nu \subseteq \rho$. Then $\forall x \in X$, $\mu(x) \le \nu(x) \le \rho(x)$. Thus $\forall x \in X$, $I(\mu(x), \nu(x)) = 1$, $I(\mu(x), \rho(x)) = 1$, and $I(\nu(x), \rho(x))) = 1$ since I satisfies the ordering principle. Hence $I(\mu, \nu) = 1$. Now $\forall x \in X$, $I(\rho(x), \mu(x)) \le I(\nu(x), \mu(x))$, and $I(\rho(x), \mu(x)) \le I(\rho(x), \nu(x))$ since I is hybrid monotonous. Now $E_I(\mu, \rho) = 1 \wedge (\wedge\{I(\rho(x), \mu(x))|x \in X\}) \le 1 \wedge (\wedge\{I(\nu(x), \mu(x))|x \in X\}) = E_I(\mu, \nu)$ and $E_I(\mu, \rho) = 1 \wedge (\wedge\{I(\rho(x), \mu(x))|x \in X\}) \le 1 \wedge (\wedge\{I(\rho(x), \nu(x)|x \in X\}) = E_I(\nu, \rho)$. ■

In [76], the following definition was used for defining fuzzy similarity measures from implication operators.

Recall the following. Let I be an implication operator. Define $S_I : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0, 1]$ by for all $(\mu, \nu) \in \mathcal{FP}(X) \times \mathcal{FP}(X)$, $S_I(\mu, \nu) = \frac{1}{n}\sum_{x\in X} I((\mu(x), \nu(x)) \wedge I((\nu(x), \mu(x)))$.

Then S_I is called a **degree of likeness**.

In [76, Theorem 2.7], it was shown that the function S of Section 7.1 is a fuzzy similarity measure.

Other implication operators can be found in [9].

We determine the smallest value a fuzzy similarity measure can be with respect to rankings since then the ratio $(S - \min)/(\max - \min)$ provides a similarity measure that ranges from 0 to 1.

For $E_I(\mu_A, \mu_B)$, (4) of Definition 1.3.1 holds vacuously for rankings A and B since $E_I(\mu_A, \mu_B)$ is never 0. (There does not exist $x \in X$ such that $\mu_A(x) = 0$ or $\mu_B(x) = 0$.)

For two rankings A and B of X, $\sum_{x \in X}(A(x) + B(x)) = n(n+1)$ and so $\sum_{x \in X}(\mu_A(x) + \mu_B(x)) = n + 1$. Thus for S of Section 7.1,

$$S(\mu_A, \mu_B) = 1 - \frac{\sum_{x \in X} |\mu_A(x) - \nu_B(x)|}{n+1}.$$

7.2 Godel and Goguen implication operators

Recall that I_1 and I_2 below are defined in Example 6.1.2.

Theorem 7.2.1. (1) *Suppose n is even. Let A be the ranking:* $1, 2, ..., \frac{n}{2}, \frac{n+2}{2}, ..., n-1, n$ *and let B be the ranking* $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, ..., 2, 1$. *Then* $I_2(\mu_A, \mu_B) = I_1(\mu_A, \mu_B) + \frac{1}{n}((n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}) - (\frac{1}{8} + \frac{1}{2n^2})$.

(2) *Suppose n is odd. Let A be the ranking* $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ *and B be the ranking* $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. *Then* $I_2(\mu_A, \mu_B) = I_1(\mu_A, \mu_B) + \frac{1}{n}((n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}) - (\frac{1}{8} - \frac{1}{8n^2})$.

Proof. (1) We have that

$$\begin{aligned} I_1(\mu_A, \mu_B) &= \frac{1}{n}(1 + ... + 1 + \frac{1}{n}(\frac{n}{2} + ... + 2 + 1)), \\ I_2(\mu_A, \mu_B) &= \frac{1}{n}(1 + ... + 1 + (\frac{\frac{n}{2}}{\frac{n+2}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})) \\ &= \frac{1}{n}(1 + ... + 1 + (\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n})). \end{aligned}$$

Hence

$$\begin{aligned} I_2(\mu_A, \mu_B) = I_1(\mu_A, \mu_B) &+ \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n}) \\ &- \frac{1}{n^2}(\frac{n}{2} + ... + 2 + 1). \end{aligned}$$

Thus

$$\begin{aligned} I_2(\mu_A, \mu_B) = I_1(\mu_A, \mu_B) &+ \frac{1}{n}(\frac{\frac{n}{2}}{\frac{n}{2}+1} + ... + \frac{2}{n-1} + \frac{1}{n}) \\ &- \frac{1}{n^2}((\frac{n}{2})(\frac{n}{2} + 1)\frac{1}{2}) \end{aligned}$$

$$= I_1(\mu_A, \mu_B) + \frac{1}{n}\sum_{i=1}^{\frac{n}{2}} \frac{i}{n-i+1} - (\frac{1}{8} + \frac{1}{2n^2})$$

Let $j = n - i + 1$. Then $i = n - j + 1$ and $j = n, n-1, ..., \frac{n}{2} + 1$. Now

$$\sum_{i=1}^{\frac{n}{2}} \frac{i}{n-i+1} = \sum_{j=\frac{n}{2}+1}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n}{2}+1}^{n} (\frac{n}{j} - 1 + \frac{1}{j})$$
$$= (n+1)(\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}) - \frac{n}{2}. \tag{7.1}$$

(2) We have that

$$I_1(\mu_A, \mu_B) = \frac{1}{n}(1 + ... + 1 + \frac{1}{n}(\frac{n-1}{2} + ... + 2 + 1)),$$
$$I_2(\mu_A, \mu_B) = \frac{1}{n}(1 + ... + 1 + (\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})).$$

Thus

$$I_2(\mu_A, \mu_B) = I_1(\mu_A, \mu_B) + \frac{1}{n}(\frac{\frac{n-1}{2}}{\frac{n+3}{2}} + ... + \frac{2}{n-1} + \frac{1}{n})$$
$$- \frac{1}{n^2}(\frac{n-1}{2} + ... + 2 + 1)$$
$$= I_1(\mu_A, \mu_B) + \frac{1}{n}\sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1} - \frac{1}{n^2}(\frac{n-1}{2})(\frac{n-1}{2} + 1)\frac{1}{2})$$
$$= I_1(\mu_A, \mu_B) + \frac{1}{n}\sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1} - (\frac{1}{8} - \frac{1}{8n^2})$$

Let $j = n - i + 1$. Then $i = n - j + 1$ and $j = n, n-1, ..., \frac{n}{2} + \frac{3}{2}$. Now

$$\sum_{i=1}^{\frac{n-1}{2}} \frac{i}{n-i+1} = \sum_{j=\frac{n+3}{2}}^{n} \frac{n-j+1}{j} = \sum_{j=\frac{n+3}{2}}^{n} (\frac{n}{j} - 1 + \frac{1}{j})$$
$$= (n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}. \tag{7.2}$$

■

We next determine approximate values for $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$ when n is even and $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$ when n is odd. These approximate values appear in the proof of the following theorem. We recall that $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is a harmonic sum which sums approximately to $\gamma + \ln n$, where γ is the Euler-Mascheroni constant, $\gamma \approx 0.5772$, where $\approx$ denotes approximately equal to.

Theorem 7.2.2. (1) *Suppose n is even. Let A be the ranking: $1, 2, ..., \frac{n}{2}, \frac{n+2}{2}, ..., n-1, n$ and let B be the ranking $n, n-1, ..., \frac{n+2}{2}, \frac{n}{2}, ..., 2, 1$. Then $I_2(\mu_A, \mu_B) \approx I_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} + \frac{1}{n} \ln 2 - \frac{1}{2n^2}$.*

(2) *Suppose n is odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B be the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Then $I_2(\mu_A, \mu_B) \approx I_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} + \ln \frac{n}{n+1} + \frac{1}{n} \ln \frac{2n}{n+1} + \frac{1}{2n} + \frac{1}{8n^2}$.*

Proof. (1) Let n be even. Consider $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n}{2}) = \ln n - \ln \frac{n}{2} = \ln 2$. Thus by Theorem 7.2.1 (1) and Eq. (7.1),

$$I_2(\mu_A, \mu_B) \approx I_1(\mu_A, \mu_B) + \frac{1}{n}((n+1)\ln 2 - \frac{n}{2}) - (\frac{1}{8} + \frac{1}{2n^2})$$
$$= I_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} + \frac{1}{n} \ln 2 - \frac{1}{2n^2}.$$

(2) Let n be odd. Consider $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}$. We have $\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{\frac{n+1}{2}} \frac{1}{j} \approx \gamma + \ln n - (\gamma + \ln \frac{n+1}{2}) = \ln n - \ln \frac{n+1}{2} = \ln \frac{2n}{n+1}$. Thus by Theorem 7.2.1 (2) and Eq. (7.2),

$$I_2(\mu_A, \mu_{B\_}) = I_1(\mu_A, \mu_B) + \frac{1}{n}((n+1)(\sum_{j=\frac{n+3}{2}}^{n} \frac{1}{j}) - \frac{n-1}{2}) - (\frac{1}{8} - \frac{1}{8n^2})$$
$$\approx I_1(\mu_A, \mu_B) + \frac{1}{n}((n+1)\ln \frac{2n}{n+1} - \frac{n-1}{2}) - (\frac{1}{8} - \frac{1}{8n^2})$$
$$= I_1(\mu_A, \mu_B) + (1 + \frac{1}{n}) \ln \frac{2n}{n+1} - \frac{1}{2} + \frac{1}{2n} - \frac{1}{8} + \frac{1}{8n^2}$$
$$= I_1(\mu_A, \mu_B) + \ln \frac{2n}{n+1} + \frac{1}{n} \ln \frac{2n}{n+1} - \frac{5}{8} + \frac{1}{2n} + \frac{1}{8n^2}$$
$$= I_1(\mu_A, \mu_B) + \ln 2 - \frac{5}{8} + \ln \frac{n}{n+1} + \frac{1}{n} \ln \frac{2n}{n+1} + \frac{1}{2n} + \frac{1}{8n^2}. \blacksquare$$

Example 7.2.3. Consider S_1. Let $n = 3$. Let A be the ranking $1, 2, 3$, and B be the ranking $3, 2, 1$. Then

$$S_1(\mu_A, \mu_B) = \frac{1}{3}(I_1(\frac{1}{3}, \frac{3}{3}) \wedge I_1(\frac{3}{3}, \frac{1}{3}) + I_1(\frac{2}{3}, \frac{2}{3}) \wedge I_1(\frac{2}{3}, \frac{2}{3}) + I_1(\frac{3}{3}, \frac{1}{3}) \wedge I_1(\frac{1}{3}, \frac{3}{3}))$$

$$= \frac{1}{3}(\frac{1}{3} + 1 + \frac{1}{3}) = \frac{5}{9}.$$

Let C be the ranking $3, 1, 2$. Then

$$S_1(\mu_A, \mu_C) = \frac{1}{3}(I_1(\frac{1}{3}, \frac{3}{3}) \wedge I_1(\frac{3}{3}, \frac{1}{3}) + I_1(\frac{2}{3}, \frac{1}{3}) \wedge I_1(\frac{1}{3}, \frac{2}{3}) + I_1(\frac{3}{3}, \frac{2}{3}) \wedge I_1(\frac{2}{3}, \frac{3}{3}))$$
$$= \frac{1}{3}(\frac{1}{3} + \frac{1}{3} + \frac{2}{3}) = \frac{4}{9}.$$

Hence for n odd, the rankings μ_A and μ_B do not give the smallest value S_1 can be.

Example 7.2.4. Consider I_1. Let $n = 6$. Let A be the ranking $1, 2, 3, 4, 5, 6$, and let B be the ranking $6, 5, 4, 3, 2, 1$. Then $I_1(\mu_A, \mu_B) = \frac{1}{6}(1+1+1+\frac{3}{6}+\frac{2}{6}+\frac{1}{6}) = \frac{24}{36}$. Let C be the ranking $6, 5, 2, 3, 1, 4$. Then $I_1(\mu_A, \mu_C) = \frac{1}{6}(1+1+\frac{2}{6}+\frac{3}{6}+\frac{1}{6}+\frac{4}{6}) = \frac{22}{36}$. Hence even though the rankings A and B yield the smallest S_1, it is not the case that A and B yield the smallest I_1.

Theorem 7.2.5. *Let n be odd. Let A be the ranking $1, 2, ..., \frac{n+1}{2}, ..., n-1, n$ and B the ranking $n, n-1, ..., \frac{n+1}{2}, ..., 2, 1$. Let C be the ranking obtain from B by interchanging the middle and the middle plus next term. Then $S_1(\mu_A, \mu_C) = \frac{1}{4} + \frac{1}{2n} - \frac{1}{4n^2}$ is the smallest value S_1 can be.*

Proof.

$$S_1(\mu_A, \mu_C) = \frac{1}{n}(n - \frac{n-1}{2} + 2(1+2+...+\frac{n-1}{2}))\frac{1}{n}$$
$$= (1 - \frac{1}{2} + \frac{2}{n}((\frac{n-1}{2})(\frac{n+1}{2}))\frac{1}{2})\frac{1}{n}$$
$$= \frac{1}{n} - \frac{1}{2n} + \frac{1}{2n^2} + \frac{n^2-1}{4n^2}$$
$$= \frac{1}{2n} + \frac{1}{2n^2} + \frac{1}{4} - \frac{1}{4n^2}$$
$$= \frac{1}{4} + \frac{1}{2n} - \frac{1}{4n^2}.$$

This is the smallest value S_1 can be since the term $2(1+2+...+\frac{n-1}{2})$ represents the smallest element in the ranking while the $n - \frac{n-1}{2}$ term represents the next smallest. ∎

For example, let $n = 7$. Then A: $1, 2, 3, 4, 5, 6, 7$, B: $7, 6, 5, 4, 3, 2, 1$, and C: $7, 6, 5, 3, 4, 2, 1$. Now

$$S_1(\mu_A, \mu_B) = \frac{1}{7}(\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + 1 + \frac{3}{7} + \frac{2}{7} + \frac{1}{7}),$$

$$S_1(\mu_A, \mu_C) = \frac{1}{7}(\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{3}{7} + \frac{2}{7} + \frac{1}{7}).$$

The middle term becomes $\frac{n - \frac{n-1}{2}}{n}$ instead of $\frac{n}{n}$.

Theorem 7.2.6. *Let A be the ranking* $1, 2, ..., n$ *and C be the ranking* $n, 1, 2, ..., n-1$. *Then* $I_1(\mu_A, \mu_C) = \frac{1}{n}(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1})$ *is the smallest value* I_1 *can be.*

Proof. In any two μ, ν, only 1 and the values of ν can appear in $I_1(\mu, \nu)$. Now for rankings, $\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n-1}$ are the smallest. None of these can appear more than once although 1 may appear more than once. ∎

Example 7.2.7. Let A be the ranking $1, 2, 3, 4, 5, 6$ and let C be the ranking $6, 1, 2, 3, 4, 5$. Then $I_1(\mu_A, \mu_C) = \frac{1}{6}(1 + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6}) = \frac{21}{36}$. By Theorem 7.2.6, this is the smallest I_1 can be. Now $S_1(\mu_A, \mu_C) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6}) = \frac{16}{6}$. Let A be the ranking $1, 2, 3, 4, 5, 6$ and B the ranking $6, 5, 4, 3, 2, 1$. Then $S_1(\mu_A, \mu_B) = \frac{12}{36}$ is the smallest S_1 can be. Hence the smallest I_1 can be doesn't yield the smallest S_1 can be.

7.3 Women and children and similarity results

The WPS Index ranks 177 countries and economies on women's status. As the only index to bring together indicators of women's inclusion, justice and security, the WPS Index is a valuable measure of women's status that can be used to track trends, guide policymaking, and hold governments accountable for their promises to advance women's rights and opportunities, [114].

The WPS Index reveals glaring disparities around the world. All countries on the index have room for improvement, and many perform considerably better or worse on some indicators of women's status than in others [114].

The application, implementation, and interpretation of the eight Fundamental Rights of a child is guided and determined by four Guiding Principles of the Convention on the Rights of the Child; the principle of nondiscrimination, the "best interests of the child", the principle of life, the survival and development, and the principle of inclusion and participation, [90].

Right to Life: The right to life means that each child must be able to live his or her life. Children have the right not to be killed. They have the right to survive and grow up in proper conditions.

Right to Education: The right to education allows each child to receive, to enjoy a social life, and to build his or her own future. The right is essential for economic, social, and cultural development.

Right to Food: The right to food is the right of each child to eat. It is the right to not die of hunger and not to suffer from malnutrition. Every 5 s, a child dies of hunger somewhere in the world.

Right to Health: The right to health means that children must be protected against illness. They must be allowed to grow and become healthy adults. This contributes to developing an active society.

Right to Water: The right to water means children have the right to safe drinking water and proper sanitary conditions. The right to water is essential for good health, survival, and proper growth.

Right to Identity: Each child has the right to have a surname, a first name, a nationality, and to know who his or her relatives are. The right to identity also means that each child's existence and rights must be officially recognized.

Right to Freedom: The right to liberty is the child's right to express him or herself, to have opinions, to have access to information, and to participate in decisions which affect his or her life. Children also have the right to religious freedom.

Right to Protection: The right to protection is the right to live in a secure and protective environment which preserves the child's well-being. Each child has the right to be protected from all forms of mistreatment, discrimination, and exploitation.

The Gender Inequality Index (GII) is an index for the measurement of gender disparity that was introduced in the 2010 Human development report. According to United Nations Development Programme (UNDP), this index is a composite measure to quantify loss of achievement within a country due to gender inequality. It uses three dimensions to measure opportunity cost: reproductive health, empowerment, and labor market participation [37]. The Human Development Index (HDI) provides a composite measure of human development used by the UNDP [37].

We next provide the scores and rank for the region: Middle East and North Africa (Table 7.1).

Let A denote the ranking of the countries for Peace and Security, B the ranking for Rights, C the ranking for GII, and D the ranking for HDI. Let X denote the set of countries. We have $n = 17$. We next determine fuzzy similarity measures. Recall the formulas $M = \frac{S}{2-S}$ and $S_L = S + \frac{1}{n}(S-1)$. We have

$$
\begin{aligned}
S(\mu_A, \mu_B) &= 1 - \frac{\sum_{x\epsilon X} |\mu_A - \mu_B|}{\sum_{x\epsilon X}(\mu_A + \mu_B)} \\
&= 1 - \frac{56}{306} \\
&= 1 - 0.183 \\
&= 0.817.
\end{aligned}
$$

Now $\sum_{x\epsilon X} |\mu_C - \mu_D| = \frac{35}{17} = \sum_{x\epsilon X} |\mu_A - \mu_C|$ and so $S(\mu_C, -\mu_D) = S(\mu_A, \mu_C)$. We have

$$
S(\mu_C, \mu_D) = 1 - \frac{\sum_{x\epsilon X} |\mu_C - \mu_D|}{\sum_{x\epsilon X}(\mu_C + \mu_D)}
$$

Table 7.1 Women and children.

Country	GII	Peace and Security	HDI	Rights
Algeria	103/10	118/11	91/8	108/10
Bahrain	49/3	56/2	42/3	85/4
Egypt	108/11	110/9	116/13	110/12
Iran	113/14	140/14	70/7	120/14
Iraq	146/16	16/15	123/15	140/16
Jordan	109/12	92/7	102/11	82/3
Kuwait	53/4	61/3	64/6	95/7
Lebanon	96/9	128/13	92/9	91/5
Libya	56/5.5	122/12	105/12	104/9
Morocco	111/13	114/10	121/14	116/13
Oman	68/8	75/5	60/5	93/6
Qatar	43/2	80/6	45/4	80/2
Saudi Arabia	56/5.5	67/4	40/2	101/8
Syrian Arab Rep.	122/15	171/16	151/16	123/15
Tunisia	65/7	96/8	95/10	79/1
United Arab Emirates	31/1	22/1	31/1	109/11
Yemen Rep.	162/17	176/17	179 / 17	170/17

$$
\begin{aligned}
&= 1 - \frac{35}{306} \\
&= 1 - 0.114 \\
&= 0.886.
\end{aligned}
$$

Hence $M(\mu_A, \mu_B) = \frac{0.817}{2-0.817} = \frac{0.817}{1.183} = 0.691$ and $M(\mu_C, \mu_D) = \frac{0.886}{2-0.886} = \frac{0.886}{1.114} = 0.795$. Also, $S_L(\mu_A, \mu_B) = 0.817 + \frac{1}{17}(0.817 - 1) = 0.817 - 0.011 = 0.806$ and $S_L(\mu_C, \mu_D) = 0.886 + \frac{1}{17}(0.886 - 1) = 0.886 - .0.007 = 0.879$.

The smallest S can be $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{34} = 0.529$. The smallest M can be is $\frac{0.529}{2-0.529} = \frac{0.529}{1.471} = 0.360$ and the smallest S_L can be is $0.529 + \frac{1}{17}(.529 - 1) = 0.529 - 0.028 = 0.501$. Thus for S

$$
\frac{0.817 - 0.529}{1 - 0.529} = \frac{0.288}{0.471} = 0.611
$$
$$
\frac{0.886 - 0.529}{1 - 0.529} = \frac{0.357}{0.471} = 0.758
$$

For M

$$
\frac{0.691 - 0.360}{1 - 0.360} = \frac{0.331}{0.640} = 0.517
$$

$$\frac{0.795-0.360}{1-0.360}=\frac{0.435}{0.640}=0.680$$

For S_L

$$\frac{0.806-0.501}{1-0.501}=\frac{0.305}{0.499}=0.611$$
$$\frac{0.879-0.501}{1-0.501}=\frac{0.378}{0.499}=0.758$$

7.4 Exercises

1. Find the fuzzy similarity measures for A and C (See Table 7.1).

CHAPTER

Space debris and sustainability[*]

8

The following is from [124]. The space sector is in the grip of an unprecedented investment spree. Collectively, there are more and more objects into space every year. At the current rate of expansion, we risk decimating the value of space for future generations. Unless we act now, an environmental crisis will be created in space, which could hamper our efforts to tackle climate change here on earth.

The present pace of growth is unsustainable. Over the past six decades, about 11,000 satellites have been launched, of which 7,000 remain in space. But that number could swell to the hundreds of thousands by the end of this decade as companies and other nation-states in building mega-constellations in low Earth orbit (LEO). Some of these new constellations will boast tens of thousands of satellites. Each one will have an expected life of between 5 and 10 years, creating vast amounts of space debris that will clutter their own orbit and endanger anything passing through it.

The environmental dangers of such space debris are myriad, including light pollution that would hinder future scientific discovery. Just as worrying are satellite reentries from the mega-constellations, which could deposit hazardous levels of alumina into the upper atmosphere. The resulting solar radiation would have pernicious consequences for the environment. The planned mega-constellations could throttle competition and innovation too, if one country or company comes to dominate a particular orbit.

However, the smart use of space can enhance life on Earth. Satellites are reducing emissions in the aviation industry by optimizing flight paths and help container ships boost efficiency and profitability. Elsewhere, space technology helps us measure global carbon emissions more accurately, allows farmers to boost yields and feed the world's growing population more sustainably. Satellites will be essential if we are to connect the roughly three billion people who have yet to use the internet. Whole industries, from mining to retail, simply would not be able to operate without suitable communications.

Regulation is needed to address space sustainability. We discuss this in a later section.

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00016-7

8.1 Space competitiveness index

The Space Competitive Index (SCI) shows a way to compare countries in their space participation levels and investments. The overall scale reflects part of a larger report that dives deeply into participation trends within each evaluated nation. Multiple factors play into the calculated SCI score. These reported scores for each of the evaluated nations rank how well each country performs in developing, creating, and executing space programs. Countries score higher if they have more human and capital resources dedicated to their space programs, great interest within the populations, and more policies supporting space participation. Consequently, these higher-scoring countries are formidable competitors in their presence in space via satellite deployment and space exploration.

To calculate the SCI, Futron Corporation used 50 separate metrics categorized into three groups for each country. These three groups of resources are government, industry, and human capital. Each group has a separate weight for determining the overall SCI score. Government and industry each account for 40% of the total while human capital is 20%

Table 8.1 Competitiveness index.

Country	Score	Rank
United States	99.67	1
Japan	48.76	2
Russia	45.29	3
China	41.85	4
Canada	39.10	5
India	28.64	6
South Korea	15.22	7
Israel	9.30	8
Australia	5.22	9

The SCI considered the European continent as one of the 15 evaluated nations. In 2008, Europe ranked second to the United States, followed by Russia, China, India, Canada, Japan, South Korea, Israel, and Brazil, [123].

8.2 Countries dominating space

The following is from [3]. Our atmosphere is filled with more than 11,000 objects that have been launched since the foray in to space began. In [3], data from Our World in Data, breaks down the amount of objects launched into space by country over time. Objects being sent into our atmosphere include satellites, crewed spacecraft, probes, and space flight equipment. Probes and landers have helped scientists explore other planets. Satellites provide us with everyday necessities like call phone service, far reaching television signals, satellite imagery, and GPS. As of late 2021, there were about 4,852 operational satellites in orbit, 2,944 belong to the United States. Many satellites in orbit are no longer functional. In fact, there is a lot of junk in space. According to NASA, there are over 27,000 pieces of space debris in orbit.

Few countries have come close to matching either the U.S. or Russia with respect to launching objects into space. In [3], 86 countries have listed with respect to having launches belonging to them. Not all these countries have orbital launch capabilities, meaning that although the satellite in space belongs to a certain country, it was launched by another. The following table is from [3].

Table 8.2 Objects launched.

Country	Score	Rank
United States	5,534	1
Russia	3,611	2
China	731	3
UK	515	4
Japan	300	5
France	130	6
India	127	7
Germany	114	8
Canada	82	9
Luxembourg	53	10
Italy	52	11
South Korea	43	12
Brazil	39	13
Australia	36	14
Belgium	36	15
Israel	30	16
Spain	29	17
Uruguay	23	18
Indonesia	21	19
Argentina	20	20
Sweden	19	21
Mexico	18	22
Saudi Arabia	17	23
United Arab Emirates	17	24
Taiwan	17	25
Finland	17	26
Turkey	16	27
Switzerland	15	28
Thailand	14	29

8.3 Responsibility for space junk

Thousands of pieces of debris from broken-down satellites, rocket boosters and weapons tests that we have launched over the years have got stuck in orbit, creating clutter which could not only crash into active satellites we need for monitoring Earth, but also release harmful chemicals into the atmosphere as they burn up on re-entry, depleting the ozone layer and space exploration.

According to the OECD, active debris removal faces several technological, geopolitical, and economic challenges. Manufacturing and launching debris removal vehicles is expensive and if it goes wrong, there is the risk of simply creating more debris. The retrieval of debris could involve sharing potentially sensitive data about the debris object's design that could involve national security, foreign policy, intellectual property. Therefore countries would realistically be limited to removing their own satellites or those of close military allies.

The issue of space debris will need to be solved soon as companies such as Boeing Co. and SpaceX get set to launch 65,000 spacecraft into low-Earl orbit, upping the likelihood of more collisions and even further debris in the future. The following table is from [32]. It lists the number of spent rocket bodies and other pieces of debris.

Table 8.3 Quantity.

Country	Quantity	Rank
Russia	7,032	1
USA	5,216	2
China	3,854	3
France	520	4
Japan	117	5
India	114	6
ESA	60	7
UK	1	8

Let A denote the SCI ranking (Table 8.1), B the Counties Dominating Index (Table 8.2), and C the Responsibility Space Junk ranking (Table 8.3). We delete those countries that do not appear in both rankings and then rerank. For A, B, we have $n = 9$. We find that

$$M(\mu_A, \mu_B) = \frac{\sum_{x\epsilon X} \mu_A(x) \wedge \mu_B(x)}{\sum_{x\epsilon X} \mu_A(x) \vee \mu_B(x)}$$
$$= \frac{41}{90 - 41} = 0.911.$$

Now the smallest M is $\frac{n+1}{3n-1} = \frac{10}{26} = 0.385$. Hence $\frac{0.911-0.385}{1-0.385} = \frac{0.526}{0.615} = 0.855$.

Now $S(\mu_A, \mu_B) = \frac{2(0.911)}{1.911} = \frac{1.822}{1.911} = 0.953$. Now the smallest S can be $\frac{2(0.385)}{1.385} = \frac{0.770}{1.385} = 0.556$. Hence $\frac{0.953-0.556}{1-0.556} = \frac{0.397}{0.444} = 0.894$.

In both cases, the fuzzy similarity measure is very high.

We next consider the B and C rankings. Here $n = 7$. We find that

$$M(\mu_B, \mu_C) = \frac{\sum_{x \in X} \mu_B(x) \wedge \mu_C(x)}{\sum_{x \in X} \mu_B(x) \vee \mu_C(x)}$$
$$= \frac{24}{56 - 24} = 0.75.$$

Now the smallest M can be is $\frac{n+1}{3n-1} = \frac{8}{20} = 0.4$. Hence $\frac{0.75-0.4}{1-0.4} = \frac{0.35}{0.6} = 0.583$.

Now $S(\mu_A, \mu_B) = \frac{2(0.75)}{1.75} = \frac{1.50}{1.75} = 0.857$. Now the smallest S can be $\frac{2(0.4)}{1.4} = \frac{0.8}{1.4} = 0.571$. Hence $\frac{0.857-0.571}{1-0.571} = \frac{0.276}{0.429} = 0.643$.

We see that the fuzzy similarity measure is medium in both cases.

8.4 Sustainability and space

In [23], guidelines for long-term sustainability of outer space activities are developed. The committee first develops the context of the guidelines. They then provide the definition, objectives, and scope of the guidelines. In the second part of [23], the guidelines are presented and explained in detail. We present the guidelines.

Guidelines for the long-term sustainability of outer space activities.

A. Policy and regulatory framework for outer pace activities

A.1 Adopt, revise, and amend, as necessary, national regulatory frameworks for outer space activities.

A.2 Consider a number of elements when developing, revising, or amending, as necessary, national regulatory frameworks for outer space activities.

A.3 Supervise national space activities.

A.4 Ensure the equitable, rational, and efficient use of the radio frequency spectrum and the various regions used by satellites.

A.5 Enhance the practice of registering space objectB Safety of space operation.

B.1 Provide updated contact information and share information on space objects and orbital events.

B.2 Improve accuracy of orbital data on space objects and enhance the practice and utility of sharing orbital information on space objectives.

B.3 Promote the collection, sharing, and dissemination of space debris monitoring information.

B.4 Perform conjunction assessment during all orbital phases of controlled flight.

B.5 Develop practical approaches for prelaunch conjunction assessment.

B.6 Share operational space weather data and forecasts.

B.7 Develop space weather models and tools and collect established practices on the mitigation of space weather effects.

B.8 Design and operation of space objects regardless of their physical and operation characteristics.

B.9 Take measures to address risks associated with the uncontrolled reentry of space objects.

B.10 Observe measures of precaution when using sources of laser beams passing through outer space.

C. International cooperation, capacity-building awareness

C.1 Promote and facilitate international cooperation in support of the long-term sustainability of outer space activities.

C.2 Share experience related to the long-term sustainability of outer space activities and develop new procedures, as appropriate, for information exchange.

C.3 Promote and support capacity-building.

C.4 Raise awareness of space activities.

D. Scientific and technical research and development

D.1 Promote and support research into and the development of ways to support sustainable exploration and use of outer space.

D.2 Investigate and consider new measures to manage the space debris population in the long term.

These guidelines are explained in great detail in [23].

8.5 Space debris and artificial intelligence

In [60], the authors consider the use of artificial intelligence (AI) to combat the problem of space debris. They state that over the past decades, a considerable and constantly growing amount of debris is being concentrated in Outer Space, originating mainly from the fragmentation of spacecraft and launch vehicles. Once in orbit, they consist in uncontrolled and noncooperative elements, orbiting at a high speed and eventually degrading under conditions of real microgravity. The main objective remains not only to estimate their position, orientation, and speed in order to avoid collision with working space objects (such as Satellites) resulting in their damage, but also to maneuver so as to avoid the collision risk below an acceptable level. Space debris identification is therefore of paramount importance for the security of space assets and success of space missions in general.

It is concluded in [60] that the use of AI technology expands the boundaries of real-time exchange and accurate analytical processing of big datasets regarding space debris identification, location, and collision risks. These benefits can only be put to good use under a multilateral cooperation scheme of monitoring and communication/exchange of information.

The following is taken from [87]. Thousands of defunct satellites, spent rocket stages, metal shards from collisions, and other remnants of human space exploration are orbiting the Earth at breakneck speeds. In [87], tracking data from the Space-Track.org maintained by the U.S. Space Force, to help visualize just how much debris is currently orbiting the Earth while identifying the biggest contributors of this celestial clutter.

It is stated in [101] that according to the data, there are roughly 14,000 small, medium, and large debris objects floating about LEO as of May 2023. And this is not counting the millions of tiny debris fragments that are too small to be tracked. Table 8.4 provides a ranking of countries responsible for most space debris.

Table 8.4 Amount of space debris.

Country	Rank	# of Space Debris
Russia (including USSR)	1	4,521
United States	2	4,317
China	3	4,137
France	4	370
India	5	62
Japan	6	48
China / Brazil	7	25
European Space Agency	8	22
Canada	9	5
Argentina	10	1
Germany	10	1
Other		24

Let D denote the ranking of the countries with respect to the above table. If we compare this ranking with the ranking B, we see that the same four countries share the top four ranks.

In [111], it is stated that the AI revolution will transform business, government, and society. The rise of ChatGPT and the ensuing arms race between big tech companies develop their own generative AI models has led to a very public debate about how best to manage the risks of this new technology.

Table 8.5 Use of AI.

Country	Rank	Country	Rank
United States	1	Hong Kong	32
China	2	Malta	33
Singapore	3	Czech Rep.	34
United Kingdom	4	Brazil	35
Canada	5	New Zealand	36
South Korea	6	Slovenia	37
Israel	7	Hungary	38
Germany	8	Turkey	39
Switzerland	9	Iceland	40
Finland	10	Chile	41
Netherlands	11	Qatar	42
Japan	12	Lithuania	43
France	13	Malaysia	44
India	14	Greece	45
Australia	15	Indonesia	46
Denmark	16	Vietnam	47
Sweden	17	Columbia	48
Luxembourg	18	Argentina	49
Ireland	19	Slovakia	50
Austria	20	Mexico	51
Spain	21	Egypt	52
Belgium	22	Uruguay	53
Italy	23	Armenia	54
Norway	24	South Africa	55
Estonia	25	Tunisia	56
Taiwan	26	Morocco	57
Poland	27	Bahrain	58
UAE	28	Pakistan	59
Portugal	29	Sri Lanka	60
Russia	30	Nigeria	61
Saudi Arabia	31	Kenya	62

The Global AI index aims to make sense of AI in 62 countries that have chosen to invest in it. It is the first ever ranking of countries based on three pillars of analysis: investment, innovation, and implementation.

Let E denote the ranking of countries in Table 8.5. We next find the fuzzy similarity measure of rankings B and E. We delete the countries that do not appear in both rankings and the rerank. Thus $n = 27$. Let X denote the set of the remaining 27 countries. We have

$$\begin{aligned} S(\mu_B, \mu_E) &= 1 - \frac{\sum_{x \epsilon X} |\mu_B(x) - \mu_E(x)|}{\sum_{x \epsilon X} (\mu_B(x) + \mu_E(x))} \\ &= 1 - \frac{161}{756} \\ &= 1 - 0.213 \\ &= 0.787. \end{aligned}$$

The smallest S can be is $\frac{1}{2} + \frac{1}{2n} = 0.519$. Hence $\frac{0.787-0.519}{1-0.519} = \frac{0.268}{0.481} = 0.557$.

Now $M = \frac{S}{2-S}$. Thus we have

$$\begin{aligned} M(\mu_B, \mu_E) &= \frac{0.787}{2 - 0.787} \\ &= 0.649. \end{aligned}$$

The smallest M can be is $\frac{0.519}{2-0.519} = 0.350$. Hence $\frac{0.649-0.350}{1-0.359} = \frac{0.299}{0.650} = 0.458$.
We see that the fuzzy similarity measure is medium.

8.6 Space debris and cybersecurity

In [52], using space junk for cyber warfare is considered. It is stated in [52] that a satellite has a life expectancy of 5 to 30 years, and even after that, it can still be in orbit with enough propellant to cruise across space and working communications that can be reactivated. Thousands of satellites, both active and dormant, have been sent into space by various organizations and governments, with 5,000 space-borne transponders connecting with Earth. Every transmission is a potential inlet for a cyber attack. Older satellites have technology commonalities, allowing cyber-exploitation of industrial systems for control and processing. Supervisory control and data acquisition (SCADA) systems within our municipalities, facilities, infrastructure, and factories are designed and built on older technology and hardware, sometimes designed decades ago, and the software is seldom updated. These SCADA systems are seen as a strategic weakness and have received more attention in recent years from the US cyber-defense sector. Satellites may be based on 1980s hardware and technology for one simple reason; they are unlikely to be improved once launched into orbit.

8.7 Exercises

1. Determine the fuzzy similarity measures for A and C.

2. Determine which countries are highly involved in multilateral cooperation scheme of monitoring and communication/exchange of information as described in Section 10.5.

3. It is stated in [31] that space systems, ranging from satellites to mission control centers are frequently the target of cyberattacks. Use [31] to write a report concerning these principles.

CHAPTER

Telecommunications ⍟ 9

9.1 Telecommunications output and network readiness

Telecommunications, also known as telecom, is the exchange of information over large distances. It is a broad term that includes various sectors, but all include a transmitter and a receiver. The medium of signal transference can be via various means, e.g., fiber, electromagnetic fields, light, and cable.

We provide the rank of countries with respect to telecommunications output according to [110] (Table 9.1).

Table 9.1 Telecommunications output.

Country	Rank	Country	Rank
Germany	1	Ireland	18
United Kingdom	2	Finland	19
France	3	Luxembourg	20
Italy	4	Hungary	21
Spain	5	Slovakia	22
Switzerland	6	Serbia	23
Netherlands	7	Bulgaria	24
Sweden	8	Croatia	25
Belgium	9	Slovenia	26
Poland	10	Latvia	27
Norway	11	Lithuania	28
Austria	12	Estonia	29
Greece	13	Cyprus	30
Romania	14	Bosnia and Herzegovina	31
Denmark	15	Iceland	32
Portugal	16	Macedonia	33
Czech Republic	17	Liechtenstein	34

⍟ This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00017-9

Table 9.2 Network readiness.

Country	Rank	Country	Rank
United States	1	United Arab Emirates	30
Singapore	2	Italy	31
Finland	3	Malta	32
Netherlands	4	Lithuania	33
Sweden	5	Poland	34
Switzerland	6	Cyprus	35
Rep. of Korea	7	Hungary	36
Denmark	8	Latvia	37
Germany	9	Russian Federation	38
United Kingdom	10	Slovakia	39
Canada	11	Malaysia	40
Israel	12	Saudi Arabia	41
Japan	13	Thailand	42
Australia	14	Ukraine	43
France	15	Brazil	44
Norway	16	Uruguay	45
Austria	17	Qatar	46
Luxembourg	18	Turkey	47
Ireland	19	Chile	48
China	20	Greece	49
Belgium	21	Croatia	50
Estonia	22	Bahrain	51
New Zealand	23	Romania	52
Iceland	24	Bulgaria	53
Hong Kong	25	Oman	54
Spain	26	Serbia	55
Czechia	27	Vietnam	56
Portugal	28	Costa Rica	57
Slovenia	29	Kazakhstan	58

The Network Readiness Index 2023 ranks a total of 134 economies that collectively account for 95% of global gross domestic product (GDP). We list the ranking according to [24] (Table 9.2).

We next determine fuzzy similarity measures of telecommunication output and readiness index, we delete the countries that are not listed in both rankings and rerank. Let X denote the resulting set of countries. Then $n = 33$. Let A denote the ranking of the telecommunication output and let B denote the ranking of the readiness index.

Then

$$
\begin{aligned}
S(\mu_A, \mu_B) &= 1 - \frac{\sum_{x \in X} |\mu_A(x) - \mu_{B(x)}|}{\sum_{x \in X} (\mu_A(x) + \mu_{B(x)})} \\
&= 1 - \frac{226}{1122} \\
&= 1 - 0.201 \\
&= 0.799.
\end{aligned}
$$

The smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{66} = 0.515$. Thus $\frac{0.799-0.515}{1-0.515} = \frac{0.284}{0.485} = 0.586$.

Now $M = \frac{S}{2-S}$. Hence

$$
\begin{aligned}
M(\mu_A, \mu_B) &= \frac{0.799}{2 - 0.799} \\
&= \frac{0.799}{1.201} \\
&= 0.665.
\end{aligned}
$$

The smallest M can be is $\frac{0.515}{2-0.515} = \frac{0.515}{1.485} = 0.347$. Thus $\frac{0.665-0.347}{1-0.347} = \frac{0.318}{0.653} = 0.487$. We see that the fuzzy similarity measure is medium.

9.2 Internet speed

The following is from [105]. The Speediest Global Index compares internet speed from around the world on a monthly basis. Data for the Index comes from the hundreds of millions of tests taken by real people using Speediest every month. Internet measurements made with Speediest occur at the times and in the places that are most relevant to the person taking the test. Each time a test is initiated, a snapshot of what the internet looks like in that place and time is recorded. When aggregated together, these individual experiences represent the typical internet performance for a given location.

From January 1, 2019 onward, countries must have at least 300 unique user tests results for mobile or fixed broadband in the reported month to be included in the Index. Prior to January 1, 2019 countries were required to have at least 3,333 unique user test results for fixed broadband at least 670 unique test results for mobile in the reported month. Results for mobile are tested on all cellular technologies. Fixed broadband includes mobile WIFI results. Results are updated mid-month for the previous month.

We next provide the ranking of countries with respect to internet speed, [105]. For Mobile 144 counties were listed and 181 for Fixed Broadband. We list only the 144 countries listed for Mobile (Table 9.3).

Table 9.3 Speediest.

Country	Mobile Rank	Fixed Broadband Rank
UAE	1	6
Qatar	2	33
Kuwait	3	26
China	4	5
Denmark	5	10
South Korea	6	24
Norway	7	32
Netherlands	8	16
Bahrain	9	69
Iceland	10	3
Saudi Arabia	11	44
United States	12	7
Finland	13	37
Macau (SAR)	14	25
Bulgaria	15	67
Sweden	16	27
Canada	17	17
India	18	87
Luxembourg	19	31
Brunei	20	75
Croatia	21	80
France	22	8
Singapore	23	1
Switzerland	24	14
Estonia	25	59
Australia	26	95
Lithuania	27	42
Latvia	28	54
Portugal	29	21
Malaysia	30	39
North Macedonia	31	115
Austria	32	61
Maldives	33	162
Greece	34	99
Taiwan	35	22
Belgium	36	48
Uruguay	37	34
Oman	38	86
Czechia	39	77
Malta	40	38

continued on next page

Table 9.3 (*continued*)

Country	Mobile Rank	Fixed Broadband Rank
Slovenia	41	52
Hong Kong (SAR)	42	
New Zealand	43	23
Cyprus	44	68
Germany	45	53
Romania	46	12
Serbia	47	76
Montenegro	48	65
United Kingdom	49	49
Brazil	50	30
Poland	51	28
Albania	52	85
Italy	53	71
Slovakia	54	72
Japan	55	15
South Africa	56	103
Suriname	57	159
Vietnam	58	43
Azerbaijan	59	118
Hungary	60	20
Spain	61	13
Botswana	62	175
Thailand	63	11
Israel	64	18
Kosovo	65	78
Georgia	66	131
Turkey	67	111
Morocco	68	130
Kazakhstan	69	102
Chile	70	4
Moldova	71	45
Mauritius	72	98
Iran	73	152
Ireland	74	40
Cote d'Ivoire	75	91
Laos	76	121
Trinidad & Tobago	77	41
Lebanon	78	163
Armenia	79	96
Uganda	80	154
Zimbabwe	81	155

continued on next page

Table 9.3 (*continued*)

Country	Mobile Rank	Fixed Broadband Rank
Honduras	82	104
Jamaica	83	81
Ethiopia	84	169
Senegal	85	138
Costa Rica	86	50
Philippines	87	51
Ukraine	88	70
Iraq	89	122
Ecuador	90	66
Uzbekistan	91	89
Guatemala	92	100
Nigeria	93	143
Kyrgyzstan	94	94
Mexico	95	88
Tanzania	96	145
El Salvador	97	101
Cambodia	98	106
Russia	99	62
Burkina Faso	100	108
Indonesia	101	126
Argentina	102	74
Kenya	103	164
Dominican Rep.	104	114
Namibia	105	157
Egypt	106	82
Bosnia and Herzegovina	107	129
Bangladesh	108	109
Guyana	109	73
Myanmar	110	143
Tunisia	111	168
Somalia	112	151
Sri Lanka	113	133
Panama	114	29
Algeria	115	158
Jordan	116	38
Paraguay	117	48
Fiji	118	148
DR Congo	119	112
Mozambique	120	153
Peru	121	47

continued on next page

Table 9.3 (*continued*)

Country	Mobile Rank	Fixed Broadband Rank
Papua New Guinea	122	149
Zambia	123	140
Pakistan	124	150
Ghana	125	113
Nepal	126	84
Libya	127	166
Mongolia	128	79
Cameroon	129	167
Belarus	130	90
Nicaragua	131	97
Columbia	132	36
Belize	133	105
Syria	134	179
Venezuela	135	110
Angola	136	135
Bolivia	137	123
Haiti	138	119
Yemen	139	174
Tajikistan	140	132
Afghanistan	141	180
Sudan	142	170
Cuba	143	181
East Timor	144	173

Let X denote the set of 36 OECD countries. Then $n = 36$. We rerank the OECD countries. Let C denote the ranking of the OECD countries with respect to mobile and D the ranking of the OECD countries for Fixed Broadband. Then

$$\begin{aligned} S(\mu_C, \mu_D) &= 1 - \frac{\sum_{x \epsilon X} |\mu_C(x) - \mu_D(x)|}{\sum_{x \epsilon X}(\mu_C(x) + \mu_D(x))} \\ &= 1 - \frac{353}{1332} \\ &= 1 - 0.266 \\ & 0.734. \end{aligned}$$

The smallest S can be is $\frac{n/2+1}{n+1} = \frac{19}{37} = 0.514$. Hence $\frac{0.734-0.514}{1-0.514} = \frac{0.220}{0.486} = 0.453$.

Now $M = \frac{S}{2-S}$. Thus $M(\mu_C, \mu_D) = \frac{0.734}{2-0.734} = \frac{0.734}{1.266} = 0.580$. The smallest M can be is $\frac{0.514}{2-0.514} = \frac{0.514}{1.486} = 0.346$. Hence $\frac{0.580-0.346}{1-0.346} = \frac{0.234}{0.654} = 0.358$. We see that the fuzzy similarity measure is low.

9.3 Spam and scam

The following is from [119]. The purpose of [119] is to understand how spam works around the world and to use that information to better protect people, to make their services better and to build trust in communication. It is stated in [119] that the problem is large. 59.49 million Americans reported having lost money to scams between July 2020 and June 2021 with an average loss per person of 02 USD. This equated to 9.8 billion USD loss when extrapolated to the adult US population.

We list the top 20 countries affected by spam calls in 2021 according to [119] (Table 9.4).

Table 9.4 Spam calls.

Country	Rank
Brazil	1
Peru	2
Ukraine	3
India	4
Mexico	5
Indonesia	6
Chile	7
Vietnam	8
South Africa	9
Russia	10
Columbia	11
Spain	12
Ecuador	13
Turkey	14
Italy	15
Honduras	16
Costa Rica	17
Greece	18
United Arab Emirates	19
United States	20

Scamming has become a global problem in recent years, and a multibillion-dollar business as the E-commerce industry has grown. People are now transferring money via the internet rather than giving it over. We next list the top 10 most scamming countries in the world according to [98] (Table 9.5).

The following is from [6]. With the increasing use of the internet in our daily lives, the risk of falling victim to on line scams is higher than ever before. Internet scamming is a global problem, with criminals operating from various countries across the world. In [6], common types of internet scams prevalent in each country are explored and tips are provided on how to stay safe on line.

Table 9.5 Scamming.

Country	Rank
Nigeria	1
India	2
China	3
Brazil	4
Pakistan	5
Indonesia	6
Venezuela	7
South Africa	8
Philippines	9
Romania	10

We list the top 10 internet scamming countries in the world in 2023 according to [6] (Table 9.6).

Table 9.6 Scamming.

Country	Rank
Nigeria	1
Ghana	2
India	3
Indonesia	4
Philippines	5
Romania	6
Russia	7
South Africa	8
Ukraine	9
United States	10

Let E denote the ranking of countries with respect to the top 10 most scamming countries in the world according to [5] and F denotes the ranking of countries with respect to the top 10 internet scamming countries in the world according to [6]. We delete the countries not appearing in both rankings and rerank. We assume E and F are these rankings. Let X denote the set of these countries. Then $n = 6$.

Then

$$\begin{aligned} S(\mu_E, \mu_F) &= 1 - \frac{\sum_{x \in X} |\mu_E(x) - \mu_F(x)|}{\sum_{x \in X} (\mu_E(x) + \mu_F(x))} \\ &= 1 - \frac{4}{42} \\ &= 1 - 0.095 \\ &0.905. \end{aligned}$$

The smallest S can be is $\frac{n/2+1}{n+1} = \frac{4}{7} = 0.571$. Hence $\frac{0.905-0.571}{1-0.571} = \frac{0.334}{0.427} = 0.779$.

Now $M = \frac{S}{2-S}$. Thus $M(\mu_E, \mu_F) = \frac{0.905}{2-0.905} = \frac{0.905}{1.095} = 0.826$. The smallest M can be is $\frac{0.571}{2-0.571} = \frac{0.571}{1.429} = 0.400$. Hence $\frac{0.826-0.400}{1-0.400} = \frac{0.426}{0.600} = 0.710$. We see that the fuzzy similarity measure is high.

9.4 Artificial intelligence

The following is from [20]. The challenges that artificial intelligence (AI) in telecommunications can address in 2023 are the following:

Poor Network management: global traffic and the need for more network equipment are growing dramatically, resulting in more complex and costly network management.

Lack of Data Analysis: telecoms struggle to leverage the vast amounts of data collected from their massive customer bases over the years. Data may be fragmented or stored across different systems, unstructured and uncategorized, or simply incomplete and not very useful.

High Costs: Following massive investments in infrastructure and digitalization, industry analysts expect telecoms' global operating expenditures to increase by billions of dollars. Many telecoms face financial crunch and must find ways to improve their bottom lines.

Crowded Marketplace: Telecom customers are determining higher quality services and better customer experience (CX) and are known to be especially susceptible to churn when their needs are not met.

In [20], common applications of AI in the telecommunication sector are given, namely,

Network optimization

Customer service automation and virtual assistants

Predictive Maintenance

Robotic Process Automation (RPA) for Telecoms

Fraud Prevention

Revenue Growth

9.5 Exercises

1. Find the fuzzy similarity measures for other regions as done for OECD countries (Table 9.3).

2. Write a report explaining in depth the common applications of AI in the telecommunication sector.

CHAPTER

Eccentric connectivity[*] 10

This chapter depends on [54]. We focus on a very important graph index, known as eccentric connectivity index initially studied by Chemists, and then adapted into Mathematics. The fuzzy graph version has an analogous computational formula and better scope for diverse applications. Topological indices have been studied widely in topology, mathematical chemistry, and chemical graph theory. Hosoya index, Wiener index, Estrada index, Hyper-Wiener index, Randic index, Padmakar-Ivan index, Szeged index, eccentric connectivity index, and Zagreb index are all examples of topological graph indices. Eccentric connectivity index was first proposed by Indian chemists Sharma, Goswami, and Madan [103]. Later Morgan, Mukwembi, and Swart [85] adopted the index into graph theory. This chapter tries to explore the fuzzy graph version of the index.

10.1 Eccentric connectivity index

After the introduction of the index by group of mathematicians in 1997 [103], the index was taken over by chemists and mathematicians, and they studied a number of properties and invented several related applications. We provide a fuzzy graph analogue of this index in Definition 10.1.1. If $G = (\Omega, \Psi)$, we consider $\Omega(w) = 1$ unless otherwise stated. Generally, it can take any value in [0, 1].

Definition 10.1.1. [54] The **Eccentric Connectivity Index, (ECI)** of a fuzzy graph $G = (\Omega, \Psi)$ denoted by $\mathcal{EF}(G)$ is defined as $\mathcal{EF}(G) = \sum_{w \in \Omega^*} d(w)l(w)$, where $d(w)$ is the degree of w and $l(w)$ is the geodesic eccentricity of w.

Definition 10.1.2. [96] The weight of a geodesic is the sum of weight of edges present in the geodesic. The geodesic originating from w having the maximum weight is the maximum geodesic of w.

Example 10.1.3. Consider $G = (\Omega, \Psi)$ in Fig. 10.1 with $\Omega^* = \{w_1, w_2, w_3, w_4, w_5\}$ and $\Psi(w_1w_2) = 0.3$, $\Psi(w_2w_3) = 0.5$, $\Psi(w_2w_4) = 0.2$, $\Psi(w_2w_5) = 0.1$, $\Psi(w_4w_5) = 0.3$.

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00018-0

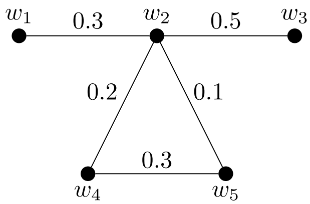

FIGURE 10.1

A fuzzy graph G with $\mathcal{EF}(G) = 2.04$.

Choose an arbitrary vertex and calculate its degree and geodesic eccentricity. For vertex w_2, the degree, $d(w_2) = 0.3 + 0.2 + 0.1 + 0.5 = 1.1$. For geodesic eccentricity, locate the weights of geodesics originating from w_2. $d_s(w_2, w_1) = 0.3$, $d_s(w_2, w_3) = 0.5$, $d_s(w_2, w_4) = 0.2$, $d_s(w_2, w_5) = 0.5$. The maximum of these values is the geodesic eccentricity. It is 0.5. Therefore $l(w_2) = 0.5$. In a similar manner, we can proceed with the remaining vertices also (Table 10.1).

Table 10.1 ECI of the fuzzy graph given in Fig. 10.1.

Vertex	$d(x)$	$l(x)$	$d(x)l(x)$
w_1	0.3	0.8	0.24
w_2	1.1	0.5	0.55
w_3	0.5	1	0.5
w_4	0.5	0.7	0.35
w_5	0.4	1	0.4
		$\mathcal{EF}(G)$	2.04

The ECI of the given fuzzy graph is, $\mathcal{EF}(G) = 0.3 \times 0.8 + 1.1 \times 0.5 + 0.5 \times 1 + 0.5 \times 0.7 + 0.4 \times 1 = 2.04$.

Next we have an observation.

Remark. Eccentric connectivity index of a fuzzy graph $G = (\Omega, \Psi)$ is zero if and only if $|\Psi^*| = 0$.

Proposition 10.1.4 provides certain bounds for the ECI of a fuzzy graph with $|\Omega^*| = n$, $|\Psi^*| = m$.

Proposition 10.1.4. *For G with $|\Omega^*| = n$, $|\Psi^*| = m$, we have*

1. $0 \leq \mathcal{EF}(G) \leq n(n-1)^2$.
2. $0 \leq \mathcal{EF}(G) \leq 2mn(n-1)$.

Proof. Consider the fuzzy graph $G = (\Omega, \Psi)$. From the definition of the index, it seems that, always $d(w) \geq 0$, $l(w) \geq 0$. Therefore $\mathcal{EF}(G) \geq 0$. When $|\Psi^*| = 0$, $d(w) = 0$ and $l(w) = 0$, which implies $\mathcal{EF}(G) = \sum_{w \in \Omega^*} d(w)l(w) = 0$. Now consider the case where $|\Omega^*| > 0$ and $|\Psi^*| > 0$. Here $d(w)$ and $l(w)$ are always greater than 0 for at least one w since there exists a strong path between any two vertices. The maximum value $d(w)$ can attain is $n-1$, it is the case when w is adjacent to all other vertices and the weight of those edges are all one. There is a possibility that for all $w \in \Omega^*$, $d(w)$ can be $n-1$. Now consider $l(w)$, the maximum weight of geodesics from w can be $n-1$, where the graph is a path. But except in the case where the graph is a complete graph with $n=2$ and the weight of each edge is one, we cannot say that for all $w \in \Omega^*$, $l(w)$ can be $n-1$. Thus

1. $0 \leq \mathcal{EF}(G) = \sum_{w \in \Omega^*} d(w)l(w) \leq \sum_{w \in \Omega^*} (n-1)(n-1) = n(n-1)^2$.
2. $0 \leq \mathcal{EF}(G) = \sum_{w \in \Omega^*} d(w)l(w) \leq \sum_{w \in \Omega^*} d(w) \sum_{w \in \Omega^*} l(w) \leq 2m \sum_{w \in \Omega^*} (n-1) = 2mn(n-1)$. ■

Generally fuzzy graph indices of fuzzy subgraphs show smaller values in comparison with the index of the mother graph. But here paradoxically the eccentric connectivity index does not satisfy this property. This is illustrated below with the help of some examples.

Example 10.1.5. The ECI of a fuzzy graph can neither be bounded above nor be bounded below by the ECI of its partial fuzzy subgraph. This is shown using the help of some examples given below in Case 1 and Case 2.
Case 1: Let $H = (\Omega, \nu)$ in Fig. 10.2 be a subgraph of $G = (\Omega, \Psi)$ in Fig. 10.1 with $\Omega^* = \{w_1, w_2, w_3, w_4, w_5\}$ and $\nu(w_1w_2) = 0.3$, $\nu(w_2w_3) = 0.3$, $\nu(w_2w_4) = 0.2$, $\nu(w_2w_5) = 0.1$. Then ECI of G is 2.04 and ECI of H is 0.77. i.e., $\mathcal{EF}(H) < \mathcal{EF}(G)$.

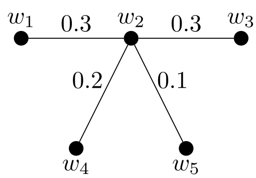

FIGURE 10.2

Fuzzy subgraph H of G with $\mathcal{EF}(H) = 0.77$.

Case 2: Consider $G = (\Omega, \Psi)$ in Fig. 10.3 with $\Omega^* = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and $\Psi(w_1w_2) = 0.1$, $\Psi(w_2w_3) = 0.09$, $\Psi(w_3w_4) = 0.09$, $\Psi(w_4w_5) = 0.1$, $\Psi(w_5w_6) = 0.09$, $\Psi(w_6w_1) = 0.09$. Let $H = (\Omega, \nu)$ in Fig. 10.4 be a subgraph of G with $\Omega^* = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and $\nu(w_2w_3) = 0.09$, $\nu(w_3w_4) = 0.09$, $\nu(w_4w_5) = 0.1$, $\nu(w_5w_6) = 0.09$, $\nu(w_6w_1) = 0.09$. Then ECI of G is 0.3136 and ECI of H is 0.3224. That is $\mathcal{EF}(H) > \mathcal{EF}(G)$.

Eccentric connectivity indices of some of the important structures are discussed in the following results. When the variable t or t_i is mentioned, its value lies between zero and one.

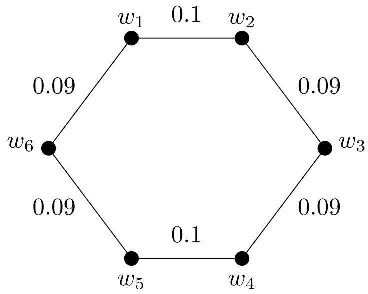

FIGURE 10.3

A fuzzy graph G with $\mathcal{EF}(G) = 0.3136$.

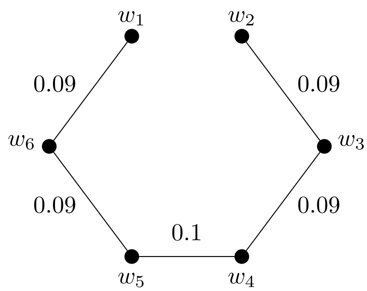

FIGURE 10.4

Partial fuzzy subgraph H of G with $\mathcal{EF}(H) = 0.3224$.

Theorem 10.1.6. *Let P be a path with $\Omega^* = \{w_1, w_2, \cdots, w_n\}$ and $\Psi(w_i w_{i+1}) = t_i$, $1 \le i \le n-1$. Then*

$$\mathcal{EF}(P) = (t_1 + t_{n-1})\left(\sum_{i=1}^{n-1} t_i\right) + \sum_{i=2}^{n-1}\left[(t_{i-1} + t_i)(\vee\{\sum_{j=1}^{i-1} t_j, \sum_{j=i}^{n-1} t_j\})\right].$$

Proof. Let P be a path. Consider vertex w_1. The degree of the vertex is t_1 since it is the only edge incident at w_1. The geodesic eccentricity of w_1 is $(\sum_{i=1}^{n-1} t_i)$, since every other geodesic is contained in this geodesic. Similarly for w_n also, i.e., $d(w_n) = t_{n-1}$ and $l(w_n) = (\sum_{i=1}^{n-1} t_i)$. Now consider w_i, $1 < i < n$, here the degree is $t_{i-1} + t_i$, since there are only two edges incident at w_i. Now the geodesic eccentricity of w_i is calculated, among all the geodesics originating from w_i, two geodesics contain all other geodesics. Therefore the geodesic which contributes maximum weight among the two is the maximum geodesic of w_i. Therefore $l(w_i) = \vee\{\sum_{j=1}^{i-1} t_j, \sum_{j=i}^{n-1} t_j\}$. Which gives the ECI of a path as $\mathcal{EF}(P) = (t_1 + t_{n-1})\left(\sum_{i=1}^{n-1} t_i\right) + \sum_{i=2}^{n-1}\Big[(t_{i-1} + t_i)(\vee\{\sum_{j=1}^{i-1} t_j, \sum_{j=i}^{n-1} t_j\})\Big]$. ■

Next result provides ECI of fuzzy cycles.

Theorem 10.1.7. *For a fuzzy cycle C with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_i w_{i+1}$ having $\Psi(e_i) = t_i$, we have*

$$\mathcal{EF}(C) = \sum_{i=1}^{n}\Big[\Big(\bigvee_{j=1}^{n}\Big\{\wedge\Big[\sum_{k=i}^{j-1} t_k, \sum_{k=j}^{i-1} t_k\Big]\Big\}\Big)(t_{i-1} + t_i)\Big]$$

where i, j are taken under modulo n.

Proof. Consider a fuzzy cycle C, since it is a fuzzy cycle all the edges are strong, therefore all the paths in C are geodesic paths. Throughout the proof i and j are taken under modulo n. Now consider an arbitrary vertex w_i, there are exactly two strong paths from w_i to each w_j. Therefore $d_s(w_i, w_j) = \wedge\Big[\sum_{k=i}^{j-1} t_k, \sum_{k=j}^{i-1} t_k\Big]$ where $1 \le j \le n$. Therefore $l(w_i) = \bigvee_{j=1}^{n}\Big\{\wedge\Big[\sum_{k=i}^{j-1} t_k, \sum_{k=j}^{i-1} t_k\Big]\Big\}$. Now the degree of vertex w_i, $d(w_i) = t_{i-1} + t_i$ for all $i \le i \le n$. Therefore $\mathcal{EF}(C) = \sum_{i=1}^{n}\Big[\Big(\bigvee_{j=1}^{n}\Big\{\wedge\Big[\sum_{k=i}^{j-1} t_k, \sum_{k=j}^{i-1} t_k\Big]\Big\}\Big)(t_{i-1} + t_i)\Big]$. ■

The following theorem finds the ECI of a CFG.

Theorem 10.1.8. *Let $G = (\Omega, \Psi)$ be a CFG with $\Omega^* = \{w_1, w_2, \cdots, w_n\}$ such that $t_1 \le t_2 \le \cdots \le t_n$, where $t_i = \Omega(w_i)$, $1 \le i \le n$. Then*

$$\mathcal{EF}(G) = (n-1)t_1^2 + \sum_{i=2}^{n-1}\{[(n-i)t_i + \sum_{j=1}^{i-1} t_j]l(w_i)\} + l(w_{n-1})\sum_{i=1}^{n-1} t_i,$$

where

$$l(w_i) = \begin{cases} 2t_1 & : \textit{if} \quad 2t_1 \le t_i \\ t_i & : \textit{if} \quad 2t_1 \ge t_i. \end{cases}$$

Proof. Let $G = (\Omega, \Psi)$ be a complete fuzzy graph. By Theorem 1.5.7, all edges of G are strong. The geodesic eccentricity of w_1 is t_1, since all geodesic paths originating from w_1, which eventually are the edges originating from w_1 having membership value t_1. For each vertex w_i, $1 < i < n$, there may exist two kinds of geodesics from w_i to w_j for $1 \leq i \neq j \leq n$. First one is the edge from w_i to w_j, which has weight t_i, other one is the path $w_i - w_1 - w_j$, which has weight $2t_1$. Now maximum geodesic is found by analyzing whether t_i or $2t_1$ is greater. So $l(w_i)$ can be written as

$$l(w_i) = \begin{cases} 2t_1 & : \text{if} \quad 2t_1 \leq t_i \\ t_i & : \text{if} \quad 2t_1 \geq t_i \end{cases}$$

For the vertex w_n, its geodesic eccentricity is same as that of the geodesic eccentricity of w_{n-1}, since its geodesic path with maximum weight is same as that of w_{n-1}. Now the degree of vertices, for w_i $1 \leq i \leq n$, it has $n - i$ edges of weight t_i incident at w_i and remaining $(n-1)-(n-i) = i-1$ edges has weight t_1 to t_{i-1}, by construction, i.e., $d(w_i) = (n-i)t_i + \sum_{j=1}^{i-1} t_j$. Therefore the ECI of a complete fuzzy graph is given by,

$$\mathcal{EF}(G) = (n-1)t_1^2 + \sum_{i=2}^{n-1}\{[(n-i)t_i + \sum_{j=1}^{i-1} t_j]l(w_i)\} + l(w_{n-1})\sum_{i=1}^{n-1} t_i, \text{ where}$$

$$l(w_i) = \begin{cases} 2t_1 & : \text{if} \quad 2t_1 \leq t_i \\ t_i & : \text{if} \quad 2t_1 \geq t_i. \end{cases} \qquad \blacksquare$$

Theorem 10.1.9. *Let $S = (\Omega, \Psi)$ be a star, with vertex set $\{w_1, w_2, \cdots, w_n\}$, where w_n is the central vertex and edge set $\{e_1, e_2, \cdots, e_{n-1}\}$ where $e_i = w_i w_n$ with $\Psi(e_i) = t_i$ satisfying $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$. Then*

$$\mathcal{EF}(S) = \Big[\sum_{i=1}^{n-1} t_i^2\Big] + t_{n-1}^2 + t_{n-1}\Big(2\Big[\sum_{i=1}^{n-2} t_i\Big] + t_{n-2}\Big).$$

Proof. Let S be a star as stated in the theorem. Let w_i be an arbitrary vertex other than the central vertex. Then the degree of w_i is t_i. The maximum geodesic of w_i is the path having edges e_i and the edge with maximum weight where $1 \leq i \leq n-2$, Therefore $l(w_i) = t_i + t_{n-1}$. Now consider w_{n-1}. Then the maximum geodesic is e_{n-1} and the edge having second last highest weight. Therefore $l(w_{n-1}) = t_{n-1} + t_{n-2}$. Next consider w_n, the central vertex. The degree, $d(w_n) = \sum_{i=1}^{n-1} t_i$. The maximum geodesic is the edge with maximum weight. Therefore $l(w_i) = t_{n-1}$. Summing up all the findings, we get that $\mathcal{EF}(S) = (\sum_{i=1}^{n-2}[t_i \times (t_i + t_{n-1})]) + t_{n-1}(t_{n-1} + t_{n-2}) + (\sum_{i=1}^{n-1} t_i) \times t_{n-1} = \Big[\sum_{i=1}^{n-1} t_i^2\Big] + t_{n-1}^2 + t_{n-1}\Big(2\Big[\sum_{i=1}^{n-2} t_i\Big] + t_{n-2}\Big)$. $\blacksquare$

Theorem 10.1.10 gives relationship between ECI of a fuzzy tree and its unique maximum spanning tree.

Theorem 10.1.10. *Let G be a fuzzy tree and F be its maximum spanning tree.*

1. *If G^* is not a tree, then $\mathcal{EF}(F) < \mathcal{EF}(G)$.*
2. *If G^* is a tree, then $\mathcal{EF}(F) = \mathcal{EF}(G)$.*

Proof. Let G be a fuzzy tree, then it has a unique maximum spanning tree. The strong edges of G are those present in the unique maximum spanning tree. Therefore $d_{s_F}(w,m) = d_{s_G}(w,m)$ for all $w, m \in \Omega^*$, which implies $l_F(w) = l_G(w)$ for all $w \in \Omega^*$. Now consider the degree of an arbitrary vertex w. When G^* is not a tree, there exist edges which are present in G and absent in F. Therefore $d_F(w) < d_G(w)$ for at least one w. When G^* is a tree, there does not exist edges which are present in G and absent in F. Therefore $d_F(w) = d_G(w)$ for every $w \in \Omega^*$. So

1. If G^* is not a tree,
$$\mathcal{EF}(F) = \sum_{w\in\Omega^*} d_F(w) l_F(w) = \sum_{w\in\Omega^*} d_F(w) l_G(w) < \sum_{w\in\Omega^*} d_G(w) l_G(w) = \mathcal{EF}(G).$$
2. If G^* is a tree,
$$\mathcal{EF}(F) = \sum_{w\in\Omega^*} d_F(w) l_F(w) = \sum_{w\in\Omega^*} d_F(w) l_G(w) = \sum_{w\in\Omega^*} d_G(w) l_G(w) = \mathcal{EF}(G).$$

■

Certain strong edges play very important role in the evaluation of ECI of a fuzzy graph as seen from the result given below.

Theorem 10.1.11. *Let $G = (\Omega, \Psi)$ be a fuzzy graph with each edge being strong. For $s, t \in \Omega^*$, let $P_{s,t}$ denote the geodesic between s and t. Suppose that wm is not a part of any $P_{s,t}$ for $s, t \in \Omega^*$ with $\{s,t\} \neq \{w,m\}$. Then*

1. *If $d_s^G(w,m) \leq \underset{j=w,m}{\wedge} \{\vee\{d_s^G(j,i); i \in \Omega^*\}\}$, then $\mathcal{EF}(G \setminus wm) = \mathcal{EF}(G) - \Psi(wm)[l_G(w) + l_G(m)]$.*
2. *If $d_s^G(w,m) > \underset{j=w,m}{\wedge} \{\vee\{d_s^G(j,i); i \in \Omega^*\}\}$, then $\mathcal{EF}(G \setminus wm) > \mathcal{EF}(G) - \Psi(wm)[l_G(w) + l_G(m)]$.*

Proof. Let G be a fuzzy graph with each edge being strong. Let wm be the strong edge that is not a part of any other geodesics. Therefore $G \setminus wm$ is also a fuzzy graph with each edge strong. The degree of all vertices other than w and m are same in G and $G \setminus wm$, i.e., $d_G(i) = d_{G\setminus wm}(i)$ for all $i \in \Omega^* \setminus \{w,m\}$. Now, $d_{G\setminus wm}(w) = d_G(w) - \Psi(wm)$. Similarly, $d_{G\setminus wm}(m) = d_G(m) - \Psi(wm)$. The geodesic eccentricity of all vertices other than w and m are same in G and $G \setminus wm$, since wm does not lie on any of the maximum geodesic of those vertices, i.e., $l_G(i) = l_{G\setminus wm}(i)$ for all $i \in \Omega^* \setminus \{w,m\}$. If $d_s^G(w,m) \leq \vee\{d_s^G(w,i); i \in \Omega^*\}$, then the geodesic eccentricity of w is same in G and $G \setminus wm$ since wm does not lie on any of the maximum geodesic of those vertices. Similarly for m also. If $d_s^G(w,m) > \vee\{d_s^G(w,i); i \in \Omega^*\}$, then the geodesic eccentricity of w changes to $\vee\{d_s^G(w,i); i \in \Omega^*\}$. Similarly for m also, i.e., $l(m) = \vee\{d_s^G(m,i); i \in \Omega^*\}$ when $d_s^G(w,m) > \vee\{d_s^G(m,i); i \in \Omega^*\}$. Now,

1. If $d_s^G(w,m) \leq \underset{j=w,m}{\wedge}\{\vee\{d_s^G(j,i); i \in \Omega^*\}\}$,

$$\begin{aligned}
&\mathcal{EF}(G \setminus wm)\\
&\quad= \sum_{i\in\Omega^*, i\neq w,m} d_{G\setminus wm}(i)l_{G\setminus wm}(i)\\
&\qquad+ [d_G(w) - \Psi(wm)]l_{G\setminus wm}(w) + [d_G(m) - \Psi(wm)]l_{G\setminus wm}(m)\\
&\quad= \sum_{i\in\Omega^*, i\neq w,m} d_G(i)l_G(i) + [d_G(w) - \Psi(wm)]l_G(w) + [d_G(m) - \Psi(wm)]l_G(m)\\
&\quad= \sum_{i\in\Omega^*, i\neq w,m} d_G(i)l_G(i) + d_G(w)l_G(w)\\
&\qquad- \Psi(wm)l_G(w) + d_G(m)l_G(m) - \Psi(wm)l_G(m)\\
&\quad= \mathcal{EF}(G) - \Psi(wm)[l_G(w) + l_G(m)],
\end{aligned}$$

i.e., $\mathcal{EF}(G \setminus wm) = \mathcal{EF}(G) - \Psi(wm)[l_G(w) + l_G(m)]$.

2. If $d_s^G(w,m) > \underset{j=w,m}{\wedge}\{\vee\{d_s^G(j,i); i \in \Omega^*\}\}$,

$$\begin{aligned}
&\mathcal{EF}(G \setminus wm)\\
&\quad= \sum_{i\in\Omega^*, i\neq w,m} d_{G\setminus wm}(i)l_{G\setminus wm}(i) + [d_G(w) - \Psi(wm)]l_{G\setminus wm}(w)\\
&\qquad+ [d_G(m) - \Psi(wm)]l_{G\setminus wm}(m)\\
&\quad= \sum_{i\in\Omega^*, i\neq w,m} d_G(i)l_G(i) + [d_G(w) - \Psi(wm)][\vee\{d_s^G(w,i); i \in \Omega^*\}]\\
&\qquad+ [d_G(m) - \Psi(wm)][\vee\{d_s^G(m,i); i \in \Omega^*\}]\\
&\quad> \sum_{i\in\Omega^*, i\neq w,m} d_G(i)l_G(i) + d_G(w)l_G(w)\\
&\qquad- \Psi(wm)l_G(w) + d_G(m)l_G(m) - \Psi(wm)l_G(m)\\
&\quad= \mathcal{EF}(G) - \Psi(wm)[l_G(w) + l_G(m)].
\end{aligned}$$

i.e., $\mathcal{EF}(G \setminus wm) > \mathcal{EF}(G) - \Psi(wm)[l_G(w) + l_G(m)]$. ■

For a saturated fuzzy cycle, $G = (\Omega, \Psi)$, $|\Omega^*| = 2t$. Hence we have the following result.

Theorem 10.1.12. *Let C be a saturated fuzzy cycle with $|\Omega^*| = n$. If every α-strong edge has strength t and every β-strong edge has strength s, then*

$$\mathcal{EF}(C) = n(s+t)(\lceil \tfrac{n}{4} \rceil s + \lfloor \tfrac{n}{4} \rfloor t).$$

Proof. Let C be a saturated fuzzy cycle. Consider an arbitrary vertex w_i. Since the graph is a saturated fuzzy cycle, it has an α-strong edge and a β-strong edge incident at the vertex w_i. Thus the degree of w_i is $s+t$. While analyzing the geodesics originating from w_i, it seems that the maximum geodesic is the geodesic ending at $w_{i+\frac{n}{2}}$, where i taken under modulo n. There are two types of such geodesics depending on whether $\frac{n}{2}$ is even or odd, which is shown in Fig. 10.5 and Fig. 10.6. The first kind has geodesic with edge weights $s, t, s, t, \cdots, t, s$ and the second kind has geodesic with edge weights $s, t, s, t, \cdots, s, t$. In both cases the geodesic has $\lceil \frac{n}{4} \rceil$ number of edges with strength s and $\lfloor \frac{n}{4} \rfloor$ number of edges with strength t. Therefore geodesic eccentricity of w_i is $\lceil \frac{n}{4} \rceil s + \lfloor \frac{n}{4} \rfloor t$. Therefore $\mathcal{EF}(C) = \sum_{w_i \in \Omega^*} (s+t)(\lceil \frac{n}{4} \rceil s + \lfloor \frac{n}{4} \rfloor t) = n(s+t)(\lceil \frac{n}{4} \rceil s + \lfloor \frac{n}{4} \rfloor t)$. ■

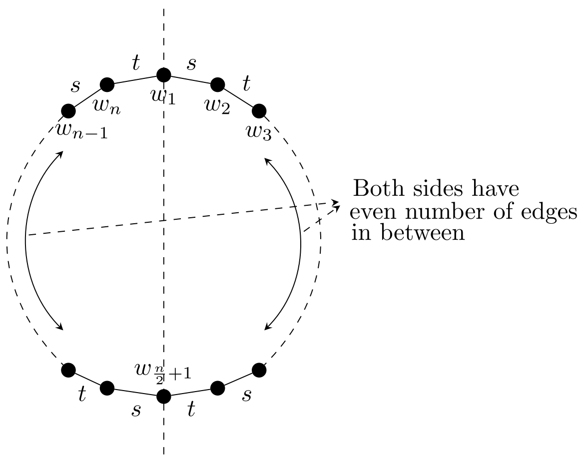

FIGURE 10.5

Saturated fuzzy cycle G with $\frac{|\Omega^*|}{2}$ being even.

Theorem 10.1.13. *For isomorphic graphs G_1 and G_2, ECI of the two graphs are equal.*

Proof. Let G_1 and G_2 be two isomorphic graphs. Let k be a bijection from G_1 to G_2. Since isomorphism preserves weight of edges and vertices, $\Omega_{G_1}(w) = \Omega_{G_2}(k(w))$ and $\Psi(wm) = \Psi(k(w)k(m))$ for $w, m \in \Omega^*$. Consider an arbitrary vertex w. Degree of the vertex w, $d_{G_1}(w) = \sum_{m \in \Omega^*} \Psi_{G_1}(wm) = \sum_{m \in \Omega^*} \Psi_{G_2}(k(w)k(m)) = d_{G_2}(w)$, i.e., the degree of a vertex is preserved. Now we find the geodesic eccentricity of w. Since the weight of the edge is preserved, the weight of geodesic is also preserved, which

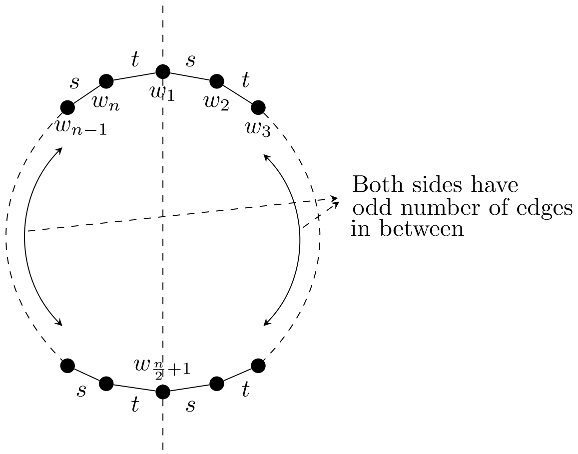

FIGURE 10.6

Saturated fuzzy cycle G with $\frac{|\Omega^*|}{2}$ being odd.

eventually shows that weight of maximum geodesic of w is also preserved. Therefore, $\mathcal{EF}(G_1) = \sum_{m\in\Omega^*} d_{G_1}(w) l_{G_1}(w) = \sum_{m\in\Omega^*} d_{G_2}(k(w)) l_{G_2}(k(w)) = \mathcal{EF}(G_2)$. ■

ECI of complement of a fuzzy graph is discussed in Theorem 10.1.14. We use the complement notion by Sunitha et al. in the theorem [108].

Theorem 10.1.14. *Consider a fuzzy cycle $G = (\Omega, \Psi)$ with $|\Omega^*| = n \geq 5$ and $\Omega(w_i) = t$ for all $w_i \in \Omega^*$. Then $\mathcal{EF}(G) + \binom{n-1}{2}\mathcal{EF}(G^c) \leq n[(n-1)t]^2$, where $G^c = (\Omega^c, \Psi^c)$ is the complement of the fuzzy graph $G = (\Omega, \Psi)$.*

Proof. Suppose $G = (\Omega, \Psi)$ be a fuzzy cycle. Let w be an arbitrary vertex. Then $d(w) = a + b$ where a and b are weight of the edges incident at w. The geodesic eccentricity of w, $l(w) \leq (n-1)t$ since the maximum geodesic of w_i is a path of length $n-1$ with each edge having weight t is the maximum possibility for $l(w)$. Therefore the ECI,

$$\mathcal{EF}(G) \leq \sum_{w\in\Omega^*} (a+b)(n-1)t \tag{10.1}$$

Next we consider the complement of the fuzzy graph G, $G^c = (\Omega^c, \Psi^c)$. Here G^c has two types of edges. First type has edges with membership value t and the second has edges with membership value $t - \Psi(e)$, where e is an edge in G and $\Psi(e) \neq 0, t$. But all edges which are of the second type are δ-edges in G^c. Now consider the same arbitrary vertex w mentioned above. By definition of degree, $d(w) = (n-3)t + (t-a) + (t-b) = (n-1)t - (a+b)$. Now consider the geodesic eccentricity of w,

$l(w) = 2t$. Since for $n \geq 5$ there exist strong paths of length less than or equal to two, between each vertex and each edge of this strong path has weight t. Therefore

$$\mathcal{EF}(G^c) = \sum_{w \in \Omega^*} [(n-1)t - (a+b)]2t = \sum_{w \in \Omega^*} [2(n-1)t^2 - (a+b)2t] \quad (10.2)$$

From Eqs. (10.1) and (10.2), $\mathcal{EF}(G) + \left(\frac{n-1}{2}\right)\mathcal{EF}(G^c) \leq \sum_{w \in \Omega^*} (a+b)(n-1)t + \sum_{w \in \Omega^*} \{[(n-1)t]^2 - (a+b)(n-1)t\} = \sum_{w \in \Omega^*} [(n-1)t]^2 = n[(n-1)t]^2$. ■

There always exist a fuzzy graph with a given ECI. It is proved in Theorem 10.1.15.

Theorem 10.1.15. *For a given $x \in \mathbb{R}^+$ there exists a fuzzy graph $G = (\Omega, \Psi)$ of ECI x.*

Proof. The result is proved by constructing such a fuzzy graph with $|\Omega^*| = n$. Now choose n and r such that $x \leq 2rn$, $0 \leq r \leq n-1$, $n \leq 2r$. Now construct a fuzzy graph such that each vertex has r edges incident on it. Let $\Omega(w)$ be greater than $\left(\frac{x}{2rn}\right)^{\frac{1}{2}}$ and for all $w \in \Omega^*$ and $\Psi(wm) = \left(\frac{x}{2rn}\right)^{\frac{1}{2}}$ for all $w, m \in \Omega^*$. Now we can calculate the ECI of the above constructed graph. Consider an arbitrary vertex w. The degree, $d(w) = r\left(\frac{x}{2rn}\right)^{\frac{1}{2}}$. The geodesic eccentricity, $l(w) = 2\left(\frac{x}{2rn}\right)^{\frac{1}{2}}$. Therefore ECI,

$$\mathcal{EF}(G) = \sum_{w \in \Omega^*} r\left(\frac{x}{2rn}\right)^{\frac{1}{2}} 2\left(\frac{x}{2rn}\right)^{\frac{1}{2}} = \sum_{w \in \Omega^*} \frac{2rx}{2rn} = \sum_{w \in \Omega^*} \frac{x}{n} = x.$$ ■

Theorem 10.1.15 guarantees the existence of a fuzzy graph having a fixed ECI, but this existence need not be unique. We can find another fuzzy graph with the same ECI having different values for Ω.

Actually we obtain a regular graph if we proceed as per the construction in Theorem 10.1.15. We can find $r-$ regular graphs of order n if and only if at least one of r and n is even, where $0 \leq r \leq n-1$. Harary graphs are examples of such regular graphs. Construction of Harary graphs is discussed through the following example.

Example 10.1.16. Let $n = 8$, $x = 10$. By Theorem 10.1.15, there exists a fuzzy graph $G = (\Omega, \Psi)$ with ECI 10. Take $|\Omega^*|$ as 8 and r as 5. Clearly, $10 \leq 2 \times 5 \times 8 = 80$, $0 \leq 5 \leq 8 - 1 = 7$, $8 \leq 2 \times 5 = 10$. So, let $\Omega^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ with $\Omega(w_1) = 0.4$, $\Omega(w_2) = 0.5$, $\Omega(w_3) = 0.4$, $\Omega(w_4) = 0.6$, $\Omega(w_5) = 0.5$, $\Omega(w_6) = 0.7$, $\Omega(w_7) = 0.6$, $\Omega(w_8) = 0.4$ and $\Psi(w_1w_2) = \Psi(w_1w_3) = \Psi(w_1w_7) = \Psi(w_1w_8) = \Psi(w_2w_3) = \Psi(w_2w_4) = \Psi(w_2w_8) = \Psi(w_3w_4) = \Psi(w_3w_5) = \Psi(w_4w_5) = \Psi(w_4w_6) = \Psi(w_5w_6) = \Psi(w_5w_7) = \Psi(w_6w_7) = \Psi(w_6w_8) = \Psi(w_7w_8) = 0.354$. Now check the ECI of G. $\mathcal{EF}(G) = \sum_{w \in \Omega^*} d(w)l(w) = \sum_{w \in \Omega^*} (5 \times 0.354) \times (2 \times 0.354) = 8 \times 1.25 = 10$ (Fig. 10.7).

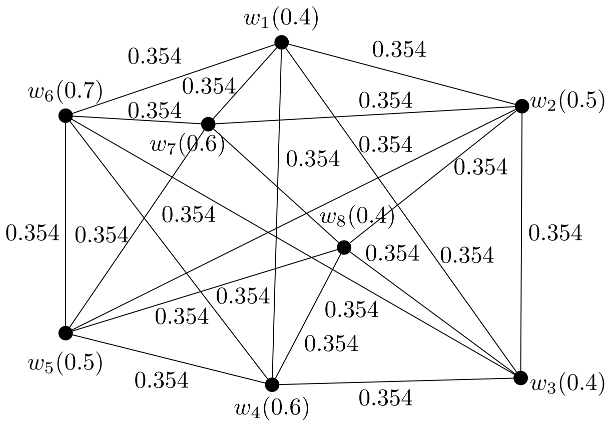

FIGURE 10.7

Fuzzy graph G with $\mathcal{EF}(G) = 10$ having eight vertices.

10.2 Modified eccentric connectivity index

Instead of degree of vertices, we can also use the strong degree of vertices in fuzzy graphs yielding stronger results. When we change the usual degree to strong degree, we get a new type of ECI as in Definition 10.2.1.

Definition 10.2.1. The **Modified Eccentric Connectivity Index (MECI)** of a fuzzy graph $G = (\Omega, \Psi)$ denoted by $\mathcal{EF}^*(G)$ is defined as $\mathcal{EF}^*(G) = \sum_{w \in \Omega^*} d_s(w) l(w)$, where $d_s(w)$ is the strong degree of w and $l(w)$ is the geodesic eccentricity of w.

Example 10.2.2. Consider Example 10.1.3. In MECI, strong degree is considered instead of usual degree in fuzzy graph, and geodesic eccentricity considered is the same. So the geodesic eccentricity of the vertices is same as that in Example 10.1.3. The strong degree of vertices is calculated and is tabulated. Also, the geodesic eccentricity of the vertices and the resulting MECI is found. Since w_2w_5 is the only non-strong edge, change of degree causes only to the vertices incident at w_2w_5 (Table 10.2).

Table 10.2 Calculating MECI of fuzzy graph given in Fig. 10.1.

Vertex	$d(x)$	$l(x)$	$d(x)l(x)$
w_1	0.3	0.8	0.24
w_2	1.0	0.5	0.5
w_3	0.5	1.0	0.5
w_4	0.5	0.7	0.35
w_5	0.3	1.0	0.3
		$\mathcal{EF}^*(G)$	1.89

After calculations, MECI of the fuzzy graph is found as 1.89.

Proposition 10.2.3. *For a fuzzy graph G* $\mathcal{EF}^*(G) \leq \mathcal{EF}(G)$.

Remark. It is obvious that if a fuzzy graph has only strong edges, then its MECI and ECI are the same. Paths, stars, and fuzzy cycles have only strong edges and hence they have this property. Also, a fuzzy tree whose support is a tree has the same MECI and ECI.

α-distance and β-distance, introduced by Mathew and Mathew [69] is a very useful pseudo-metric. Using this, we can modify the index discussed before to define α-eccentric connectivity index and β-eccentric connectivity index. A comparison of the values of these indices with ECI is made below.

Definition 10.2.4. The α-eccentric connectivity index ($\alpha - ECI$) of a fuzzy graph $G = (\Omega, \Psi)$ denoted by $\alpha - \mathcal{EF}(G)$ is defined as $\alpha - \mathcal{EF}(G) = \sum_{w \in \Omega^*} d_\alpha(w) l_\alpha(w)$, where $d_\alpha(w)$ is the sum of weights of α-strong edges incident at w and $l_\alpha(w)$ is the maximum α-distance from w to m over all $m \in \Omega^*$.

Definition 10.2.5. The β-eccentric connectivity index ($\beta - ECI$) of a fuzzy graph $G = (\Omega, \Psi)$ denoted by $\beta - \mathcal{EF}(G)$ is defined as $\beta - \mathcal{EF}(G) = \sum_{w \in \Omega^*} d_\beta(w) l_\beta(w)$, where $d_\beta(w)$ is the sum of weights of β-strong edges incident at w and $l_\beta(w)$ is the maximum β-distance from w to m over all $m \in \Omega^*$.

Fuzzy graphs like paths and stars behave similarly since they have similar kinds of edges. If P, is a path, then it has only α-strong edges and hence $\alpha - \mathcal{EF}(P) = \mathcal{EF}(P)$ and $\beta - \mathcal{EF}(P) = 0$. For a star S, which is not a fuzzy star, $\alpha - \mathcal{EF}(S) = \mathcal{EF}(S)$ and $\beta - \mathcal{EF}(S) = 0$, since it has only alpha strong edges.

In the case of a fuzzy tree this comparison is little difficult as it can contain δ-edges apart form α-strong edges. For a fuzzy tree G, $\alpha - \mathcal{EF}(G) \leq \mathcal{EF}(G)$ and $\beta - \mathcal{EF}(G) = 0$ because α-strong edges of G are precisely the edges present in the maximum spanning tree of G.

We can observe that α-ECI and β-ECI of two major structures; fuzzy cycle and a CFG are not generally bounded by their ECI. The case of a fuzzy cycle is given below using an example and the case of CFG is similar and is left to the reader.

Example 10.2.6. If G is a fuzzy cycle, then the α-ECI of G is neither bounded above nor bounded below by its ECI. Consider the following three cases.

Case 1: Consider a fuzzy graph $G = (\Omega, \Psi)$ with $\Omega^* = \{w_1, w_2, w_3\}$ and $\Psi(w_1 w_2) = 0.1$, $\Psi(w_1 w_3) = 0.1$, $\Psi(w_2 w_3) = 0.6$. After calculations, we can see that α-ECI of G is 0.72 and ECI of G is 0.3. Therefore α-ECI of G > ECI of G. The case has depicted in Figs. 10.8 and 10.9.

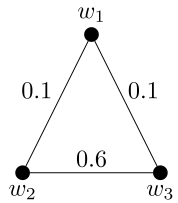

FIGURE 10.8

A fuzzy graph G with $\mathcal{EF}(G) = 0.3$.

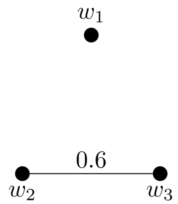

FIGURE 10.9

Fuzzy subgraph of G in Fig. 10.9 with α-strong edges only, whose $\alpha - \mathcal{EF}(G) = 0.72$.

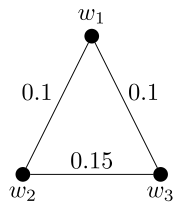

FIGURE 10.10

A fuzzy graph G with $\mathcal{EF}(G) = 0.095$.

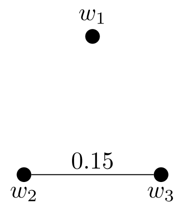

FIGURE 10.11

Fuzzy subgraph of G in Fig. 10.9 with α-strong edges only, whose $\alpha - \mathcal{EF}(G) = 0.045$.

Case 2: Consider a fuzzy graph $G = (\Omega, \Psi)$ with $\Omega^* = \{w_1, w_2, w_3\}$ and $\Psi(w_1 w_2) = 0.1$, $\Psi(w_1 w_3) = 0.1$, $\Psi(w_2 w_3) = 0.15$. After calculations we can see that $\alpha-$ ECI of G is 0.045 and ECI of G is 0.095. Therefore $\alpha-$ ECI of $G <$ ECI of G. The case has depicted in Figs. 10.10 and 10.11.

Case 3: A fuzzy cycle has at least two β-strong edges. Therefore α-ECI of G is never equal to the ECI of G.

The case of β-ECI of a fuzzy cycle G is similar.

Proposition 10.2.7. *The β-ECI of a CFG can be greater, equal or lesser than its ECI.*

Proof. Consider a complete fuzzy graph G as stated in Theorem 10.1.8. By Theorem 1.5.9, G has at most one α-strong edge. If G has no α-strong edges, then $\beta - \mathcal{EF}(G) = \mathcal{EF}(G)$. If it has an α-strong edge, then $\beta - \mathcal{EF}(G) = \mathcal{EF}(G) - [d(w_n)l(w_n) + d(w_{n-1})l(w_{n-1})] + 2(d(w_{n-1} - t_{n-1})l_\beta(w_{n-1}) \cdots (1)$

Case 1: $2t_{k-1} \leq t_{n-1}$, $1 \leq k < n-1$.
Then $-[d(w_n)l(w_n) + d(w_{n-1})l(w_{n-1})] + 2(d(w_{n-1} - t_{n-1})2t_1$ in equation (1) becomes, $-2\Big(\sum_{i=1}^{n-1} t_i\Big)2t_1 + 2\Big(\sum_{i=1}^{n-2} t_i\Big)2t_1 < 0$. Therefore $\beta - \mathcal{EF}(G) > \mathcal{EF}(G)$.

Case 2: $2t_{k-1} \geq t_{n-1}$, $1 \leq k < n-1$.
Then $-[d(w_n)l(w_n) + d(w_{n-1})l(w_{n-1})] + 2(d(w_{n-1} - t_{n-1})2t_1$ in equation (1) becomes, $-2\Big(\sum_{i=1}^{n-1} t_i\Big)2t_{n-1} + 2\Big(\sum_{i=1}^{n-2} t_i\Big)2t_1$, which can be positive, negative or equal to zero. Therefore $\beta-$ ECI of a complete fuzzy graph can be greater, equal or lesser than the ECI of a complete fuzzy graph. ∎

Similarly, we can see that α-ECI of a CFG G is neither bounded above nor bounded below by ECI of the CFG G.

10.3 Algorithm

We will discuss an algorithm for determining the ECI of a fuzzy graph in this section.

Algorithm. Let $G = (\Omega, \Psi)$ be a fuzzy graph with n vertices. Let $\Omega^* = \{w_1, w_2, \cdots, w_n\}$.

1. Construct the matrix $A = [a_{ij}]$ with $a_{ij} = \Psi(w_i w_j)$.
2. Calculate $d(i)$, where $d(i) = \sum_{j=1}^{n} a_{ij}$.
3. Using algorithm in [62] find α-strong, β-strong, δ-edges.
4. Obtain a subgraph $G' = (\Omega', \Psi')$ of G having only alpha and beta strong edges.
5. Using Dijkstra's Algorithm in [26] find the shortest path between all vertices in G'. Name the shortest path between w_i and w_j by P_{ij}.

6. Let SP_{ij} be the sum of weight of edges in P_{ij}.
7. Construct an $n \times n$ matrix L corresponding to $G = (\Omega, \Psi)$ with the following properties. Each row and column corresponds to vertices in Ω^*. If row i corresponds to vertex w_i and column j corresponds to vertex w_j, then SP_{ij} is the entry corresponds to row i and column j.
8. Let $l(i)$ be the largest membership value of the i^{th} row of matrix L.
9. Then $\mathcal{EF}(G) = \sum_{i=1}^{n} d(i) \times l(i)$.

Illustration of Algorithm: Let $G = (\Omega, \Psi)$ be a fuzzy graph in Fig. 10.12 with $\Omega^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ and $\Psi(w_1w_2) = 0.3$, $\Psi(w_1w_3) = 0.4$, $\Psi(w_2w_4) = 0.5$, $\Psi(w_2w_6) = 0.3$, $\Psi(w_3w_5) = 0.1$, $\Psi(w_3w_6) = 0.4$, $\Psi(w_4w_6) = 0.2$, $\Psi(w_5w_6) = 0.4$, $\Psi(w_6w_7) = 0.1$, $\Psi(w_6w_8) = 0.3$.

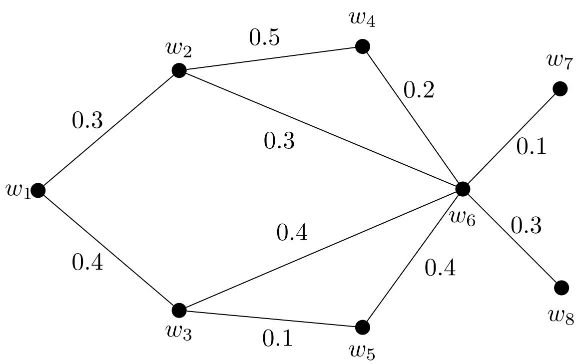

FIGURE 10.12

Fuzzy graph for the illustration of Algorithm.

Step 1: The matrix corresponding to the given fuzzy graph is

$$A = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{array} \\ \begin{pmatrix} 0 & 0.3 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0.5 & 0 & 0.3 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0.1 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.2 & 0.4 & 0 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 \end{pmatrix} \end{array}$$

Step 2: Calculate $d(i)$ of different vertices. Towards this, take the sum of each row in matrix discussed in step 1 (Table 10.3).

Table 10.3 Degree of vertices.

Vertex	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$d(vertex)$	0.7	1.1	0.9	0.7	0.5	1.7	0.1	0.3

Step 3: Using algorithm in [62] finds α-strong, β-strong, δ-edges (Table 10.4).

Table 10.4 α-strong, β-strong, and δ-edges of the fuzzy graph.

α-strong edges	β-strong edges	δ-edges
w_1w_3, w_2w_4, w_3w_6, w_5w_6, w_6w_7, w_6w_8	w_1w_2, w_2w_6	w_3w_5, w_4w_6

Step 4: The sub-graph having only alpha and beta strong edges is plotted in Fig. 10.13.

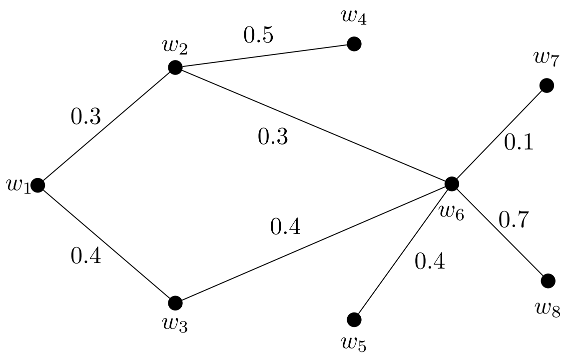

FIGURE 10.13

Subgraph of fuzzy graph in Fig. 10.13.

Step 5, 6, and 7: Dijkstra's algorithm is used and have found the shortest paths between each vertex. They are plotted in Table 10.3, in such a way that, each cell contains the shortest path between vertices given in the column and row head along with their weight.

Step 8: Calculate $l(i)$ of different vertices. Given in Table 10.5, see also Table 10.6.

Step 9: $\mathcal{EF}(G) = 0.7 \times 1 + 1.1 \times 0.7 + 0.9 \times 1.2 + 0.7 \times 1.2 + 0.5 \times 1 + 1.7 \times 0.8 + 0.1 \times 0.9 + 0.3 \times 1.1 = 5.67.$

10.4 Application

Human trafficking has taken different forms in the modern era. It includes sex trade, forced labor, domestic servitude and indecent factory work. The victims generally

Table 10.5 Shortest path and its weight between vertices.

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	–	w_1w_2 0.3	w_1w_3 0.4	$w_1w_2w_4$ 0.8	$w_1w_2w_6w_5$ 1	$w_1w_2w_6$ 0.6	$w_1w_2w_6w_7$ 0.7	$w_1w_2w_6w_8$ 0.9
w_2	w_1w_2 0.3	–	$w_2w_1w_3$ 0.7	w_2w_4 0.5	$w_2w_6w_5$ 0.7	w_2w_6 0.3	$w_2w_6w_7$ 0.4	$w_2w_6w_8$ 0.6
w_3	w_1w_3 0.4	$w_2w_1w_3$ 0.7	–	$w_3w_1w_2w_4$ 1.2	$w_3w_6w_5$ 0.8	w_3w_6 0.4	$w_3w_6w_7$ 0.5	$w_3w_6w_8$ 0.7
w_4	$w_1w_2w_4$ 0.8	w_2w_4 0.5	$w_3w_1w_2w_4$ 1.2	–	$w_4w_2w_6w_5$ 1.2	$w_4w_2w_6$ 0.8	$w_4w_2w_6w_7$ 0.9	$w_4w_2w_6w_8$ 1.1
w_5	$w_1w_2w_6w_5$ 1	$w_2w_6w_5$ 0.7	$w_3w_6w_5$ 0.8	$w_4w_2w_6w_5$ 1.2	–	w_5w_6 0.4	$w_5w_6w_7$ 0.5	$w_5w_6w_8$ 0.7
w_6	$w_1w_2w_6$ 0.6	w_2w_6 0.3	w_3w_6 0.4	$w_4w_2w_6$ 0.8	w_5w_6 0.4	–	w_6w_7 0.1	w_6w_8 0.3
w_7	$w_1w_2w_6w_7$ 0.7	$w_2w_6w_7$ 0.4	$w_3w_6w_7$ 0.5	$w_4w_2w_6w_7$ 0.9	$w_5w_6w_7$ 0.5	w_6w_7 0.1	–	$w_7w_6w_8$ 0.4
w_8	$w_1w_2w_6w_8$ 0.9	$w_2w_6w_8$ 0.6	$w_3w_6w_8$ 0.7	$w_4w_2w_6w_8$ 1.1	$w_5w_6w_8$ 0.7	w_6w_8 0.3	$w_7w_6w_8$ 0.4	–

Table 10.6 Geodesic eccentricity of vertices.

Vertex	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$l(vertex)$	1	0.7	1.2	1.2	1	0.8	0.9	1.1

suffer from psychological and even physical abuses, beatings, sexual slavery and abuses, food and sleep deprivation, threats to family members, and severe isolation from the world outside [1].

A study on how governments consider and combat modern slavery was undertaken in [40]. 161 countries were considered and government responses were recorded. The responses involve (1) survivors supported, (2) criminal justice, (3) coordination and accountability, (4) addressing risk, and (5) government and business. It can be seen in [40].

By statistical testing the authors grouped 24 countries of vulnerability into four different dimensions of covering: (1) civil and political protections, (2) social health and economic rights, (3) personal security, and (4) refugee populations and conflict. In [40], we provide tables for giving measures of vulnerability to modern slavery by countries with respect to these four dimensions.

A normalization of the data was carried out of government responses and vulnerability ratings. We take $\otimes$ as a t-norm and $\oplus$ as a t-conorm. σ denotes the government response of country ratings and μ the success of combating modern slavery with respect to the edge xy, where x and y are countries and $\mu(xy) = \sigma(x) \otimes \sigma(y)$. τ represents the vulnerability ratings of different countries. We define ν to be $\nu(xy) = \tau(x) \oplus \tau(y)$. We consider ν as giving a measure of failure in fighting human slavery with regard to the edge xy. The tables in [40] provide large numbers when the vulnerability of a country is very high. Consequently, the complements provide large numbers if when the vulnerability is very low. Hence we put more interest in the vulnerability ratings complements because they provide high government responses. ν^c represent the standard complement of ν. In the following, we let $\otimes$ denote product and $\oplus$ denote algebraic sum, i.e., $a \oplus b = a + b - a \otimes b$.

We present a list of countries involved with trafficking to the United States of America through different routes. The routes are through South America. These routes are given below and are discussed in [91].

China → Columbia → Guatemala → Mexico → US
India → Guatemala → Mexico → US
Ethiopia → S. Africa → Brazil → Ecuador → Mexico → US
Somalia → UAE → Russia → Cuba → Columbia → Mexico → US
Nigeria → Spain → Cuba → Columbia → Mexico → US
Nigeria → Spain → Columbia → Mexico → US

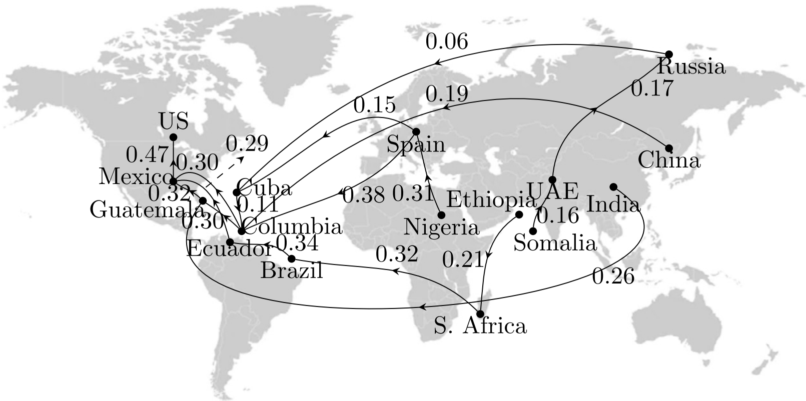

FIGURE 10.14

Routes to the US: Mapping human smuggling networks.

In the following, we merge these routes into one directed graph (Fig. 10.14). Our analysis does not depend on the direction involved between countries. Consequently, we consider the graph to be undirected (Table 10.7).

Table 10.7 Geodesic eccentricity of vertices.

Country	σ	τ^c	Edge	μ	ν^c
China	0.36	0.55	(China, Columbia)	0.19	0.32
Columbia	0.53	0.58	(Columbia, Guatemala)	0.30	0.34
			(Columbia, Mexico)	0.30	0.33
Guatemala	0.56	0.58	(Guatemala, Mexico)	0.32	0.33
Mexico	0.57	0.57	(Mexico, US)	0.47	0.47
India	0.82	0.82	(India, Guatemala)	0.26	0.27
Ethiopia	0.46	0.47	(Ethiopia, S. Africa)	0.21	0.21
S. Africa	0.42	0.42	(S. Africa. Brazil)	0.32	0.35
Brazil	0.49	0.51	(Brazil, Ecuador)	0.34	0.45
Ecuador	0.66	0.69	(Ecuador, Mexico)	0.29	0.37
Somalia	0.51	0.65	(Somalia, UAE)	0.16	0.21
UAE	0.28	0.28	(UAE, Russia)	0.17	0.43
Russia	0.56	0.74	(Russia, Cuba)	0.06	0.39
Cuba	0.30	0.58	(Cuba, Columbia)	0.11	0.39
Nigeria	0.44	0.44	(Nigeria, Spain)	0.31	9.35
Spain	0.71	0.80	(Spain, Cuba)	0.15	0.54
			(Spain, Columbia)	0.38	0.39

We next determine the d_s values for the source countries China, India, Ethiopia, Somalia, Nigeria, and the destination country, United States.

Note that between China and Somalia and between China and Nigeria, there are two strong paths each.

The d_s values are determined as follows: d_s(China, India) = μ(China, Columbia) $+\mu$ (Columbia, Guatemala)
$+\mu$(Guatemala, India) = σ(China) $\otimes$ σ(Columbia) + σ(Columbia) $\otimes$ σ(Guatemala) + σ(Guatemala) $\otimes$ σ(India) = 0.75.

We obtain the following d_s values in a similar manner.

d_s(China, India) = 0.75, d_s(China, Ethiopia) = 1.97, d_s(China, Somalia) = 0.69, d_s(China, Nigeria) = 0.88,
d_s(India, Ethiopia) = 1.74, d_s(India, Somalia) = 1.38, d_s(India, Nigeria) = 1.57,
d_s(Ethiopia, Somalia) = 1.96, d_s(Ethiopia, Nigeria) = 2.15, d_s(Somalia, Nigeria) = 1.19
d_s(China, US) = 1.28,
d_s(India, US) = 1.05,
d_s(Ethiopia, US) = 1.76,
d_s(Somalia, US) = 1.27,
d_s(Nigeria, US) = 1.46.

We next determine the vertex degree of some key countries. The vertices have three or more adjacent edges.
d(Columbia) = 0.1.28,
d(Guatemala) = 0.86,
d(Mexico) = 1.38,
d(Cuba) = 0.32,
d(Spain) = 0.84.

Low government response leads to a higher susceptibility to modern slavery. We next determine the average of the d_s values above taken over the number of edges involved.

ad_s(China, India) = 0.75/3 = 0.25, ad_s(China, Ethiopia) = 1.97/7 = 0.28, ad_s(China, Somalia) = 0.69/5 = 0.14, ad_s(China, Nigeria) = 0.88/3 = 0.29,
ad_s(India, Ethiopia) = 1.74/6 = 0.29, ad_s(India, Somalia) = 1.38/7 = 0.20, ad_s(India, Nigeria) = 1.57/5 = 0.31,
ad_s(Ethiopia, Somalia) = 1.96/9 = 0.22, ad_s(Ethiopia, Nigeria) = 2.15/6 = 0.36, ad_s(Somalia, Nigeria)= 1.19/6 = 0.20.

We see that the lowest average government response is the route between China and Somalia.
ad_s(China, US) = 1.28/4 = 0.32,
ad_s(India, US) = 1.05/3 = 0.35,
ad_s(Ethiopia, US) = 1.76/5 = 0.35,
ad_s(Somalia, US) = 1.27/6 = 0.22,
ad_s(Nigeria, US) = 1.46/4 = 0.36.

We see that lowest average government response is the route to the US from Somalia.

We next determine the average vertex degree value.
$ad(\text{Columbia}) = 0.1.28/5 = 0.26$,
$ad(\text{Guatemala}) = 0.86/3 = 0.29$,
$ad(\text{Mexico}) = 1.38/4 = 0.34$,
$ad(\text{Cuba}) = 0.32/3 = 0.11$,
$ad(\text{Spain}) = 0.84/3 = 0.28$.
The lowest average response involves Cuba.

We next consider the vulnerability of countries with respect to modern slavery. Recall that $\nu(xy) = \tau(x) \oplus \tau(y) \geq \tau(x) \vee \tau(y)$ and

$$\begin{aligned}
\nu^c(xy) &= 1 - \nu(xy) = 1 - (\tau(x) \oplus \tau(y)) \\
&= 1 - \tau(x) - \tau(y) + \tau(x) \otimes \tau(y) \\
&= (1 - \tau(x)) \otimes (1 - \tau(y)) \\
&= \tau^c(x) \otimes \tau^c(y) \\
&\leq \tau^c(x) \wedge \tau^c(y).
\end{aligned}$$

Also, let P be a path with n edges and let v_i be respective vulnerability values of the edges, $i = 1, ..., n$. Consider $\sum_{i=1}^{n} v_i$ and the d_s sum $\sum_{i=1}^{n}(1 - v_i) = n - \sum_{i=1}^{n} v_i$. Then the average values are $\frac{1}{n}\sum_{i=1}^{n} v_i$ and $1 - \frac{1}{n}\sum_{i=1}^{n} v_i$. Consequently, we can proceed with vulnerability as we did with government response.

$d_s(\text{China, India}) = 0.93$, $d_s(\text{China, Ethiopia}) = 2.37$, $d_s(\text{China, Somalia}) = 1.74$, $d_s(\text{China, Nigeria}) = 1.06$,
$d_s(\text{India, Ethiopia}) = 2.08$, $d_s(\text{India, Somalia}) = 2.45$, $d_s(\text{India, Nigeria}) = 1.77$,
$d_s(\text{Ethiopia, Somalia}) = 3.13$, $d_s(\text{Ethiopia, Nigeria}) = 2.45$, $d_s(\text{Somalia, Nigeria}) = 1.92$,
$d_s(\text{China, US}) = 1.46$,
$d_s(\text{India, US}) = 1.17$,
$d_s(\text{Ethiopia, US}) = 1.85$,
$d_s(\text{Somalia, US}) = 2.22$,
$d_s(\text{Nigeria, US}) = 1.54$.

We next determine the vertex degree of some key countries. The vertices have three or more adjacent edges.
$d(\text{Columbia}) = 1.77$,
$d(\text{Guatemala}) = 1.04$,
$d(\text{Mexico}) = 1.50$,
$d(\text{Cuba}) = 1.32$,
$d(\text{Spain}) = 1.28$.

We next compute average values.

ad_s(China, India) $= 0.93/3 = 0.31$, ad_s(China, Ethiopia) $= 2.37/7 = 0.34$, ad_s(China, Somalia) $= 1.74/5 = 0.35$, ad_s(China, Nigeria) $= 1.06/3 = 0.35$,

ad_s(India, Ethiopia) $= 2.08/6 = 0.35$, ad_s(India, Somalia) $= 2.45/7 = 0.35$, ad_s(India, Nigeria) $= 1.77/5 = 0.35$,

ad_s(Ethiopia, Somalia) $= 3.13/9 = 0.35$, ad_s(Ethiopia, Nigeria) $= 2.45/6 = 0.34$, ad_s(Somalia, Nigeria) $= 1.92/6 = 0.33$.

We see that there is no significant difference in vulnerability to modern slavery among the various routes.

We next consider routes to the United States.
ad_s(China, US) $= 1.46/4 = 0.37$,
ad_s(India, US) $= 1.17/3 = 0.39$,
ad_s(Ethiopia, US) $= 1.85/5 = 0.37$,
ad_s(Somalia, US) $= 2.22/6 = 0.37$,
ad_s(Nigeria, US) $= 1.54/4 = 0.38$.

Once again, we find no significance in vulnerability among the routes from different origin countries to the United States.

We also have the following average vertex degree values.
ad(Columbia) $= 1.77/5 = 0.35$,
ad(Guatemala) $= 1.04/3 = 0.37$,
ad(Mexico) $= 1.50/4 = 0.38$,
ad(Cuba) $= 1.32/3 = 0.44$,
ad(Spain) $= 1.28/3 = 0.43$.

In general, the vulnerability of countries to modern slavery seems to be high. An increase in government response appears to be necessary.

10.5 Exercises

1. Calculate the eccentric connectivity index of the fuzzy graph given by $G = (\Omega, \Psi)$ with $\Omega^* = \{w_1, w_2, w_3, w_4, w_5\}$ with $\Omega(w_1) = 0.15$, $\Omega(w_2) = 0.2$, $\Omega(w_3) = 0.25$, $\Omega(w_4) = 0.8$, $\Omega(w_5) = 0.9$ and $\Psi(w_1w_2) = 0.15$, $\Psi(w_1w_3) = 0.15$, $\Psi(w_1w_4) = 0.15$, $\Psi(w_1w_5) = 0.15$, $\Psi(w_2w_3) = 0.2$, $\Psi(w_2w_4) = 0.2$, $\Psi(w_2w_5) = 0.2$, $\Psi(w_3w_4) = 0.25$, $\Psi(w_3w_5) = 0.25$, $\Psi(w_4w_5) = 0.8$.

2. Calculate the modified eccentric connectivity index of the fuzzy graph given by $G = (\Omega, \Psi)$ where $\Omega^* = \{w_1, w_2, w_3, w_4, w_5\}$ with $\Omega(w) = 1$ for every $w \in \Omega^*$ and $\Psi(w_1w_2) = 0.4$, $\Psi(w_2w_3) = 0.6$, $\Psi(w_2w_4) = 0.3$, $\Psi(w_2w_5) = 0.2$, $\Psi(w_4w_5) = 0.4$.

3. Construct a fuzzy graph whose eccentric connectivity index is 12.

CHAPTER

Neighborhood connectivity index[*] 11

While developing an interconnection network in a particular area A, the information about the traffic in the surroundings of A is very helpful. Transportation networks and water connection networks are suitable examples. A new fuzzy graph index which speaks about the neighborhood flow of its vertices is discussed in this chapter. This concept is significant in modern networks and can be used in allocation problems and routing problems. This chapter heavily depends upon [53]. Basics of fuzzy graph theory can be found in [58,68,79,80,82].

11.1 Neighborhood connectivity index of a fuzzy graph

This section is all about a new connectivity index in fuzzy graphs, named neighborhood connectivity index. The strength of a vertex and strengths of all vertices in its neighborhoods play significant roles in the index. The following results will be useful.

Theorem 11.1.1. *[34] If $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \leq \mu_2$, then $\sigma_2 \geq \mu_1$ and vice versa.*

Theorem 11.1.2. *[17] For a complete fuzzy graph $CI(G) = WI(G)$.*

Definition 11.1.3. The **Neighborhood Connectivity Index** (NCI)**,** of a fuzzy graph G is defined as $NCI(G) = \sum_{m \in V(G)} d(m)e(m)$, where $d(m)$ is the cardinality of $N(m)$ and $e(m) = \vee\{\mu(mp)\colon p \in N(m)\}$ with $N(m) = \{p : \mu(mp) > 0, m, p \in \sigma^*\}$. $e(m)$ is termed as the **potential** of the vertex m.

In graph theory a vertex having highest value of potential is called **maximum potential vertex**. It is denoted by $e_G(m)$. Similarly we use $d_G(m)$, $N_G(m)$ for $d(m)$, $N(m)$ respectively in graphs.

$e(m)$ can be defined in a different way using connectivity as follows. For any vertex m, $e(m) = \vee\{CONN_G(m, p) : p \in V(G)\}$. For every $x \in \sigma^* \setminus \{m\}$, a strongest $m - x$ path P, (say) contains an edge from $E(m)$, where $E(m) = \{mp\colon p \in N(m)\}$.

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00019-2

If $\vee\{\mu(mp)\colon p \in N(m)\} = \alpha$, then the strength of P is less than or equal to α. In particular if $\mu(mz) = \alpha$ then mz is a strongest path with strength $e(m)$. Therefore both the definitions of $e(m)$ are equivalent.

Example 11.1.4. Consider $G = (\sigma, \mu)$ with $\sigma^* = \{l, a, m, b, n, c\}; \sigma(l) = 0.5$, $\sigma(a) = 0.6$, $\sigma(m) = 0.2$, $\sigma(b) = 0.7$, $\sigma(n) = 0.4$, $\sigma(c) = 0.3$, and $\mu(la) = 0.4$, $\mu(ln) = 0.1$, $\mu(lc) = 0.2$, $\mu(am) = 0.1$, $\mu(an) = 0.3$, $\mu(mn) = 0.2$, $\mu(bn) = 0.4$, $\mu(nc) = 0.3$.

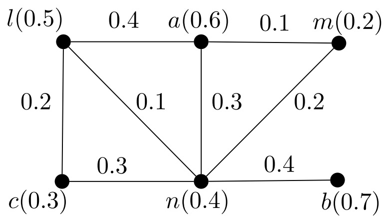

FIGURE 11.1

A fuzzy graph G with $NCI(G) = 5.8$.

We can see that $d(l) = 3$, $e(l) = \vee\{0.4, 0.2, 0.1\} = 0.4$. Similarly, we proceed with other vertices also.

Vertex	$d(x)$	$e(x)$	$d(x)e(x)$
l	3	0.4	1.2
a	3	0.4	1.2
m	2	0.2	0.4
b	1	0.4	0.4
n	5	0.4	2
c	2	0.3	0.6
		$NCI(G)$	5.8

Thus for G in Fig. 11.1, $NCI(G) = 3 \times 0.4 + 3 \times 0.4 + 2 \times 0.2 + 1 \times 0.4 + 5 \times 0.4 + 2 \times 0.3 = 5.8$.

NCI of a fuzzy graph is zero if and only if the cardinality of its edge set is zero.

Proposition 11.1.5. *If $H = (\tau, \nu)$ is a partial fuzzy subgraph of $G = (\sigma, \mu)$, then $NCI(H) \leq NCI(G)$.*

Proof. Suppose $H = (\tau, \nu)$ be a partial fuzzy subgraph of $G = (\sigma, \mu)$, with $\sigma^* = \{m_1, m_2, \cdots, m_n\}$. Let m be an arbitrary vertex in τ^*. Then $\nu(mm_i) \leq \mu(mm_i)$ for all other vertices m_i in τ^*. Therefore $\vee_i\{\nu(mm_i)\} \leq \vee_i\{\mu(mm_i)\}$. Also, $d_H(m) \leq d_G(m)$. Therefore

$$NCI(H) = \sum_{m_i} d_H(m_i) \vee_i \{\nu(mm_i)\} \leq \sum_{m_i} d_G(m_i) \vee_i \{\mu(mm_i)\} = NCI(G). \qquad \blacksquare$$

Example 11.1.6. Consider $H = (\tau, \nu)$ in Fig. 11.2. Clearly H is a partial fuzzy subgraph of $G = (\sigma, \mu)$ mentioned in Example 11.1.4. After computing the connectedness between the vertices and cardinality of neighborhood for each vertex, NCI of H can be calculated as 4.2 which is less than NCI of G, which is 5.8.

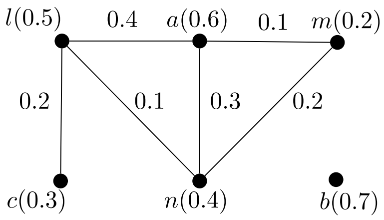

FIGURE 11.2

Subgraph H of G with $NCI(H) = 4.2$.

Now we provide some bounds for the index. From Proposition 11.1.5, Corollary 11.1.7 follows.

Corollary 11.1.7. *For a fuzzy graph $G = (\sigma, \mu)$ with vertex set σ^* and complete fuzzy super graph $G' = (\sigma', \mu')$ spanned by σ^*, we have $0 \leq NCI(G) \leq NCI(G')$.*

Proposition 11.1.8. *For G with $|\sigma^*| = n$, $0 \leq NCI(G) \leq n(n-1)$.*

Proof. Consider $G = (\sigma, \mu)$. If $\mu^* = \phi$, then $d(m) = 0$, $e(m) = 0$ for all $m \in \sigma^*$. Which implies $NCI(G) = 0$. If $|\mu^*| > 0$, then $0 < d(m) \leq n - 1$, $0 < e(m) \leq 1$ for at least one $m \in \sigma^*$. Which implies $0 < NCI(G) \leq \sum_{m \in \sigma^*} (n-1) \times 1 = n(n-1)$. The upper bound occurs when the underlying graph is a complete graph and there exist at least on edge incident to each vertex having strength 1. Therefore $0 \leq NCI(G) \leq n(n-1)$. ■

Proposition 11.1.9. *Let $G = (\sigma, \mu)$ be a connected fuzzy graph with n edges. Then $2nt \leq NCI(G) \leq 2ns$ where $t = \wedge\{e(m) : m \in \sigma^*\}$ and $s = \vee\{e(m) : m \in \sigma^*\}$.*

Proof. Suppose $G = (\sigma, \mu)$ is a fuzzy graph with n edges. Then

$$NCI(G) = \sum_{m \in \sigma^*} e(m)d(m) \leq \sum_{m \in \sigma^*} sd(m) = s \sum_{m \in \sigma^*} d(m) = s \times 2n = 2sn.$$

Similarly,

$$NCI(G) = \sum_{m \in \sigma^*} e(m)d(m) \geq \sum_{m \in \sigma^*} td(m) = t \sum_{m \in \sigma^*} d(m) = t \times 2n = 2tn.$$

Therefore $2nt \leq NCI(G) \leq 2ns$. ■

Note that Equality holds in Proposition 11.1.9 when all vertices have the same potential.

In the following corollaries, we find the NCI of structures such as trees, cycles and complete fuzzy graphs.

Corollary 11.1.10. *Consider a fuzzy graph $G = (\sigma, \mu)$ where G^* is a tree. Let $\vee\{d(m) : m \in \sigma^*\} = r$ and let S_i be the set of vertices containing all vertices with degree i, $1 \leq i \leq r$. Then $NCI(G) = \sum_{m \in S_i, i=1}^{r} i \sum_{p \in N(m)} \vee\{\mu(mp)\}$.*

Corollary 11.1.11. *For a cycle $G = (\sigma, \mu)$ having edges $e_1, e_2, \cdots, e_n$ with $\mu(e_i) = t_i$ and $t_{n+1} = t_1$ we have $NCI(G) = 2\sum_{i=1}^{n} \vee\{t_i, t_{i+1}\}$.*

Corollary 11.1.12. *Let $G = (\sigma, \mu)$ be a CFG with $\sigma^* = \{m_1, m_2, \cdots, m_n\}$ such that $t_1 \leq t_2 \leq \cdots \leq t_n$, where $t_i = \sigma(m_i)$, $1 \leq i \leq n$. Then $NCI(G) = (n-1)(t_1 + t_2 + \cdots + t_{n-2} + t_{n-1} + t_{n-1})$.*

Proof. Consider the graph G. We know that, for a CFG, $\mu(m_i m_j) > 0$ for all $m_i, m_j \in \sigma^*$. Therefore $d(m_i) = n - 1$ for all m_i, $1 \leq i \leq n$. Now, we can check the potential of vertices. While considering m_1 we see that it is the vertex with minimum membership value. So we can see that $CONN_G(m_1, m_i) = t_1$, $2 \leq i \leq n$. Therefore $e(m_1) = t_1$. Next, consider the vertices m_i, $1 < i < n$. Here, $CONN_G(m_s, m_i) \leq t_i$ for all $s < i$, $CONN_G(m_r, m_i) = t_i$ for all $r > i$; therefore $e(m_i) = t_i$, $2 \leq i \leq n-1$. At last, we consider the vertex m_n. Here we can see that $CONN_G(m_i, m_n) \leq t_{n-1}$, $1 \leq i \leq n-1$, since there is no edge of membership value t_n, and there is an edge of membership value t_{n-1}. Therefore $e(m_n) = t_{n-1}$. Summing up all those values, we get $NCI(G) = (n-1)(t_1 + t_2 + \cdots + t_{n-2} + t_{n-1} + t_{n-1})$. ■

Proposition 11.1.13. *Neighborhood connectivity index of two isomorphic fuzzy graphs are equal.*

Proof. Let j be a bijection between the isomorphic fuzzy graphs G_1 and G_2. Since weights of the edges and vertices are preserved by an isomorphism, $N_{G_1}(m) = N_{G_2}(j(m))$, which implies $d_{G_1}(m) = d_{G_2}(j(m))$ for $m \in \sigma_1^*$. Similarly, $CONN_{G_1}(m, p) = CONN_{G_2}(j(m), j(p))$ for $m, p \in \sigma_1^*$. Implying $e_{G_1}(m) = e_{G_2}(j(m))$. Therefore

$$NCI(G_1) = \sum_{m \in V(G)} d_{G_1}(m) e_{G_2}(m)$$
$$= \sum_{f(m) \in V(G)} d_{G_2}(j(m)) e_{G_2}(j(m)) = NCI(G_2).$$

i.e., $NCI(G_1) = NCI(G_2)$. ■

Theorem 11.1.14. *Consider a fuzzy graph $G = (\sigma, \mu)$. If $0 \leq t_1 \leq t_2 \leq 1$, then $NCI(G^{t_2}) \leq NCI(G^{t_1})$.*

Proof. Consider a fuzzy graph $G = (\sigma, \mu)$. In G^{t_2} number of edges with nonzero strength incident at a vertex is less than or equal to the number of edges with nonzero strength incident at a vertex in G^{t_1}. Therefore $d_{G^{t_2}}(m) \leq d_{G^{t_1}}(m)$. If $\mu_G(mp) \leq t_1$, then $\mu_{G^{t_2}}(mp) = \mu_{G^{t_1}}(mp)$. If $t_1 < \mu_G(mp) \leq t_2$, then $\mu_{G^{t_2}}(mp) \leq \mu_{G^{t_1}}(mp)$. If $\mu_G(mp) > t_2$, then $\mu_{G^{t_2}}(mp) = \mu_{G^{t_1}}(mp)$. Now for $m \in \sigma^*$, $CONN_{G^{t_2}}(m, p) \leq CONN_{G^{t_1}}(m, p)$ for all $p \in \sigma^*$. Therefore $e_{G^{t_2}}(m) \leq e_{G^{t_1}}(m)$. Therefore

$$NCI(G^{t_2}) = \sum_{m \in V(G)} e_{G^{t_2}}(m) d_{G^{t_2}}(m) \leq \sum_{m \in V(G)} e_{G^{t_1}}(m) d_{G^{t_1}}(m) = NCI(G^{t_1}).$$ ■

In Theorem 11.1.15, we discuss about saturated fuzzy cycles.

Theorem 11.1.15. *Consider a saturated fuzzy cycle G with $|V(G^*)| = n$ for which every $\alpha-$ strong edge is of strength t and every $\beta-$ strong edge is of constant strength, then $NCI(G) = 2nt$.*

Proof. Suppose $G = (\sigma, \mu)$ is as in statement of the theorem. Since G^* is a saturated fuzzy cycle, $d(m) = 2$ for any $m \in \sigma^*$. Also from the assumption it follows that t is greater than the constant strength of $\beta-$ strong edges, which implies, $e(m) = t$ for all $m \in \sigma^*$. Therefore $NCI(G) = \sum_{i=1}^{n} 2t = 2nt$. ■

Example 11.1.16. Consider the fuzzy cycle G so that $G^* = C_n$ as given in Fig. 11.3. Clearly it is a saturated fuzzy cycle with $\sigma^* = \{l, a, m, b, n, c, o, d\}$, $\mu(la) = 0.4$, $\mu(am) = 0.2$, $\mu(mb) = 0.4$, $\mu(bn) = 0.2$, $\mu(nc) = 0.4$, $\mu(co) = 0.2$, $\mu(od) = 0.4$, $\mu(dl) = 0.2$. Then neighborhood connectivity index, $NCI(G) = 2 \times 8 \times 0.4 = 6.4$.

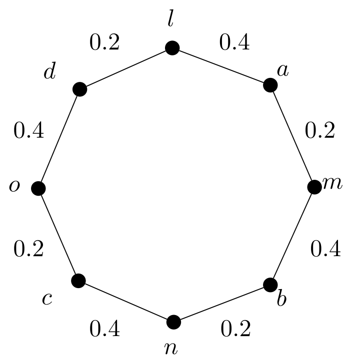

FIGURE 11.3

Saturated fuzzy cycle G with $NCI(G) = 6.4$.

Theorem 11.1.17. *There does not exist any kind of connected super fuzzy graph with equal neighborhood connectivity index as that of the parent graph.*

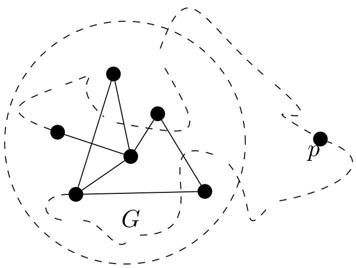

FIGURE 11.4

A fuzzy graph with an extra vertex than the parent graph.

Proof. Consider a graph H which has a vertex p in addition to the parent graph G as shown in Fig. 11.4. Let $m \in \sigma_G^*$ then $d_G(m) \leq d_H(m)$, since m may or may not have an edge with p. Now consider p, since $p \notin G$, $0 = d_G(p) < d_H(p)$. While considering the potential of the edges. For $m \in \sigma_G^*$, $e_G(m) \leq e_H(m)$, since there may or may not have an edge with strength greater than $e_G(m)$, adjacent to p. While considering p it is obvious that $0 = e_G(p) < e_H(p)$. Therefore

$$
\begin{aligned}
NCI(G) &= \sum_{m \in \sigma_G^*} d_G(m) e_G(m) \\
&= \sum_{m \in \sigma_G^*} d_G(m) e_G(m) + d_G(p) e_G(p) < \sum_{m \in \sigma_G^*} d_H(m) e_H(m) + d_H(p) e_H(p) \\
&= NCI(H).
\end{aligned}
$$

Now we have shown that there does not exist a connected super graph having same NCI as that of the parent graph when we add a vertex. Next consider a graph H, which has an edge e in addition to the parent graph G as shown in Fig. 11.5. There exists at least one vertex m in G to which e is incident, then $d_G(m) < d_H(m)$. Also, we can see that $e_G(m) \leq e_H(m)$. Therefore

$$NCI(G) = \sum_{m \in \sigma_G^*} d_G(m) e_G(m) < \sum_{m \in \sigma_G^*} d_H(m) e_H(m) = NCI(H).$$

Now we have shown that there does not exist a connected super graph having same NCI as that of the parent graph when we add an edge. ■

The following two theorems ventilate a way for construction of fuzzy graphs with a given NCI value with some predefined constraints.

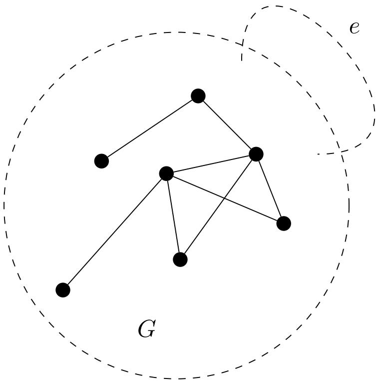

FIGURE 11.5

A fuzzy graph which has an extra edge than the parent graph.

Theorem 11.1.18. *For a given $n \in \mathbb{N}$, $x \in \mathbb{R}$ with $x \le 2n$, there exists a fuzzy graph $G = (\sigma, \mu)$ of neighborhood connectivity index x with $|\mu^*| = n$.*

Proof. Let $|\mu^*| = n$. Construct a fuzzy graph $G = (\sigma, \mu)$ such that $\sigma(m_i) \ge \frac{x}{2n}$ for all $m_i \in \sigma^*$, $\mu(m_i m_j) = \frac{x}{2n}$ for all $m_i m_j \in \mu^*$. Now we can check neighborhood connectivity index of the constructed graph. Here $e(m_i) = \frac{x}{2n}$ for all $m_i \in \sigma^*$. Therefore

$$NCI(G) = \sum_{m_i \in V(G)} d(m_i)\frac{x}{2n} = \frac{x}{2n} \sum_{m_i \in V(G)} d(m_i) = \frac{x}{2n} \times 2n = x.$$

Hence our constructed graph is a fuzzy graph of NCI x with $|\mu^*| = n$. ■

Theorem 11.1.19. *For a given $n \in \mathbb{N}$, $x \in \mathbb{R}$ with $x \le n(n-1)$, there exists a fuzzy graph $G = (\sigma, \mu)$ of neighborhood connectivity index x with $|\sigma^*| = n$.*

Proof. We can prove this theorem by similar construction from Theorem 11.1.18 by taking $|\sigma^*| = n$, $\sigma(m_i) \ge \frac{x}{n(n-1)}$ for all $m_i \in \sigma^*$ and $\mu(m_i m_j) = \frac{x}{n(n-1)}$ for all $m_i m_j \in \mu$. ■

Example 11.1.20. Let $|\mu^*| = 4$, $x = 4$. Clearly, $4 \le 8$. Now we can find a fuzzy graph $G = (\sigma, \mu)$ given in Fig. 11.6 such that $\sigma(l) = 0.8$, $\sigma(a) = 0.6$, $\sigma(m) = 0.5$, $\sigma(b) = 0.6$, $\mu(la) = 0.5$, $\mu(lm) = 0.5$, $\mu(am) = 0.5$, $\mu(ab) = 0.5$ with neighborhood connectivity index, $NCI(G) = 4$.

Proposition 11.1.21. *Consider a fuzzy cycle $G = (\sigma, \mu)$ with $|\sigma^*| = n \ge 4$ and $\sigma(m_i) = t$ for all $m_i \in \sigma^*$. Then $NCI(G^c) - NCI(G) \ge n^2 t - 5nt$, where $G^c = (\sigma^c, \mu^c)$ is the fuzzy complement of the fuzzy graph $G = (\sigma, \mu)$.*

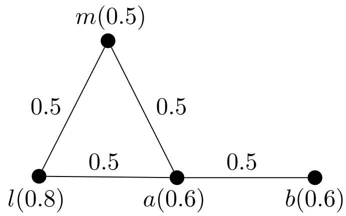

FIGURE 11.6

Fuzzy graph with $NCI(G) = 4$, given $m = 4$, $x = 4$.

Proof. Suppose $G = (\sigma, \mu)$ is a fuzzy cycle. The neighborhood of each vertex in the fuzzy cycle has two vertices. Therefore $d(m) = 2$. The potential of each vertex will always be less than t, since each vertex has strength t. Therefore $e(m) \leq t$. Therefore

$$NCI(G) = \sum_{m \in V(G)} d(m)e(m) = 2 \sum_{m \in V(G)} e(m) \leq 2nt \qquad (11.1)$$

Now consider the complement $G^c = (\sigma^c, \mu^c)$ of the graph $G = (\sigma, \mu)$. Clearly G^c will have all edges which are not present on the cycle. In addition to that some edges of the cycles can also appear. Therefore each vertex can have a neighborhood of cardinality greater than $n - 3$. i.e., $d(m) \geq n - 3$ for all $m \in \sigma$. Since all the edges other than those lying on the cycle have strength t, and all others have strength less than t, we can say $e(m) = t$. Therefore

$$NCI(G^c) = \sum_{m \in V(G)} d(m)e(m) = t \sum_{m \in V(G)} d(m) \geq nt(n-3) = n^2t - 3nt \qquad (11.2)$$

From Eqs. (11.1) and (11.2), $NCI(G^c) - NCI(G) \geq n^2t - 3nt - 2nt = n^2t - 5nt$. ∎

Theorem 11.1.22. *For a fuzzy tree G, which is not a tree, with $F = (\sigma, \nu)$ as its maximum spanning tree, $NCI(F) < NCI(G)$.*

Proof. Suppose $G = (\sigma, \mu)$ is a fuzzy tree, which is not a tree. Let $F = (\sigma, \nu)$ be the maximum spanning tree of G.

Claim: For each vertex p in G, the edge with maximum strength incident at p will also lie on the maximum spanning tree F of G.

Proof of claim: Suppose not, let p be a vertex in G and pm be the edge with maximum strength incident at p. Suppose pm does not lie on the maximum spanning tree. Then $CONN_F(p, m) < CONN_G(p, m)$, a contradiction. Hence the claim. Now consider an arbitrary vertex m, then $e(m)$ is the maximum of the weight of edges starting from m. Hence by the claim, we proved that $e_F(m) = e_G(m)$. Now we will show that $d_F(m) < d_G(m)$. Since our given fuzzy graph is not a tree, the maximum spanning tree of G will be different from G. There will be at least one edge removed from G. Let mp be such an edge. Then clearly, $d_F(m) < d_G(m)$ and $d_F(p) < d_G(p)$. Therefore

$$\begin{aligned} NCI(F) &= \sum_{m \in V(G)} d_F(m) e_F(m) = \sum_{m \in V(G)} d_F(m) e_G(m) \\ &< \sum_{m \in V(G)} d_G(m) e_G(m) = NCI(G), \end{aligned}$$

i.e., $NCI(F) < NCI(G)$. ■

Definition 11.1.23. Two sets of vertices are called a **twinning vertex sets of cardinality** r, if each set has cardinality r and neighborhood connectivity index of the graph obtained after removing each set is same.

Theorem 11.1.24. *Consider a fuzzy graph G. Let A be the set of pendant vertices with potential a. B be the set of supporting vertices of vertices from A with degree c and potential b. Then all the subgraphs obtained after removing any one vertex from the set A will have equal neighborhood connectivity index, i.e., any two subsets of cardinality one of set A are examples of twinning vertex sets of cardinality one.*

Proof. Consider a fuzzy graph G. Let A and B be as defined in the theorem statement. We show that for $u, v \in A$, $NCI(G \setminus u) = NCI(G \setminus v)$. First, we consider vertices which does not belong to B or A. Let f be such a vertex. Then clearly $d_{G\setminus u}(f) = d_{G\setminus v}(f)$ and $e_{G\setminus u}(f) = e_{G\setminus v}(f)$. Let a be the supporting vertex of u and b be the supporting vertex of v. Then $d_{G\setminus u}(a) = d_{G\setminus v}(b)$, since $d_G(a)$ and $d_G(b)$ are equal and removing a pendant vertex reduces it by one and $e_{G\setminus u}(a) = e_{G\setminus v}(b)$, by condition. For those vertices g which belong to A, but they are not u and v and those vertices belong to B but they are not a and b, $d_{G\setminus u}(g) = d_{G\setminus v}(g)$ and $e_{G\setminus u}(g) = e_{G\setminus v}(g)$. Now consider $u, v \in A$. For them we have $d_{G\setminus v}(u) = d_{G\setminus u}(v)$ and $e_{G\setminus v}(u) = e_{G\setminus u}(v)$.

Now,

$$\begin{aligned} NCI(G \setminus v) &= \sum_{t \notin B, t \notin A} d_{G\setminus v}(t) e_{G\setminus v}(t) + \sum_{t \in A, t \neq u, t \neq v} d_{G\setminus v}(t) e_{G\setminus v}(t) \\ &\quad + \sum_{t \in B, t \neq a, t \neq b} d_{G\setminus v}(t) e_{G\setminus v}(t) + d_{G\setminus v}(u) e_{G\setminus v}(u) + d_{G\setminus v}(b) e_{G\setminus v}(b) \\ &= \sum_{t \notin B, t \notin A} d_{G\setminus u}(t) e_{G\setminus u}(t) + \sum_{t \in A, t \neq u, t \neq v} d_{G\setminus u}(t) e_{G\setminus u}(t) \\ &\quad + \sum_{t \in B, t \neq a, t \neq b} d_{G\setminus u}(t) e_{G\setminus u}(t) + d_{G\setminus u}(v) e_{G\setminus u}(v) + d_{G\setminus u}(b) e_{G\setminus u}(b) \\ &= NCI(G \setminus u). \end{aligned}$$ ■

The following Corollary follows from Theorem 11.1.24.

Corollary 11.1.25. *Consider a fuzzy graph G. Let*

1. *A be the set of pendant vertices with potential a.*
2. *B be the set of supporting vertices of vertices from A with degree c and potential b.*
3. *A_i be the set of vertices of A having same supporting vertex from B.*

Then the NCI of the subgraph obtained after removing s number of vertices from any A_i will be same, i.e., such sets are twinning vertex sets of cardinality s.

Example 11.1.26. Consider the fuzzy graph G as in Fig. 11.7 with $\sigma^* = \{l, a, m, b, n, c, o, d, p, e, q, f, r\}$ and $\mu(la) = 0.3$, $\mu(am) = 0.2$, $\mu(ac) = 0.3$, $\mu(mb) = 0.2$, $\mu(md) = 0.3$, $\mu(mo) = 0.4$, $\mu(bn) = 0.4$, $\mu(bd) = 0.1$, $\mu(bq) = 0.2$, $\mu(bf) = 0.1$, $\mu(pq) = 0.3$, $\mu(eq) = 0.3$, $\mu(fr) = 0.4$.

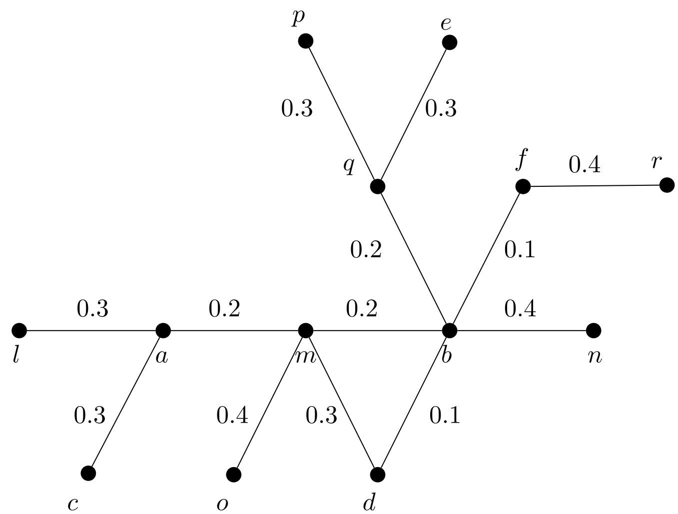

FIGURE 11.7

Fuzzy graph having twinning vertex sets.

Here $NCI(G) = 8.9$, $NCI(G \setminus l) = 7.9$, $NCI(G \setminus p) = 7.9$, $NCI(G \setminus \{l, c\}) = 7.2$, $NCI(G \setminus \{p, e\}) = 7.2$. It shows that $\{l\}$ and $\{p\}$ are twinning vertex sets of cardinality one and $\{l, c\}$ and $\{p, e\}$ are twinning vertex sets of cardinality two.

The remaining section compares NCI with connectivity index and Wiener index.

Theorem 11.1.27. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph, $CI(G)$ be the connectivity index of G and $NCI(G)$ the neighborhood connectivity index of G. Then $2CI(G) \leq NCI(G)$.*

Proof. Let $G = (\sigma, \mu)$ be a complete fuzzy graph. Then

$$2CI(G) = 2 \sum_{m,p\in\sigma^*} \sigma(m)\sigma(p)CONN_G(m,p) \le 2 \sum_{m,p\in\sigma^*} CONN_G(m,p),$$

(since $0 < \sigma(m), \sigma(p) \le 1$) $\le \sum_{m,p\in\sigma^*} e(m) + \sum_{m,p\in\sigma^*} e(p)$, (replace one $CONN_G(m,p)$ with $e(m)$ and another with $e(p)$) $= NCI(G)$, since each $e(m)$ repeats $d(m)$ times altogether. ■

Remark. For a complete fuzzy graph we have $2WI(G) \le NCI(G)$ since $CI(G) = WI(G)$ by Theorem 11.1.2.

The above result is not always true. Consider the fuzzy graph $G = (\sigma, \mu)$ given in Fig. 11.8 such that $\sigma^* = \{l, a, m, b, n\}$, $\mu(la) = 0.1$, $\mu(am) = 0.2$, $\mu(an) = 0.3$, $\mu(mn) = 0.1$, $\mu(bn) = 0.3$. The neighborhood connectivity index, $NCI(G) = 2.6$ and the connectivity index, $CI(G) = 1.9$. Here $2CI(G) = 3.8 \nleq 2.6 = NCI(G)$. Also note that Wiener index, $WI(G) = 4.2$ which is greater than 2.6.

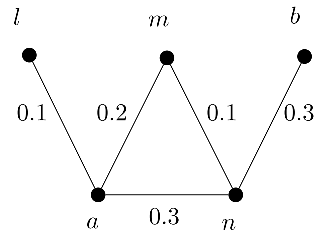

FIGURE 11.8

Fuzzy graph with $2CI(G) > NCI(G)$.

11.2 Fuzzy graph operations

There are several fuzzy graph operations in fuzzy graph theory. This section deals about the NCI of graphs obtained by some of these operations. As defined earlier, in this section $G_1 \cup G_2$ represents union, $G_1 + G_2$ represents join, $G_1[G_2]$ represents composition, $G_1 \times G_2$ represents Cartesian product and $G_1 \otimes G_2$ represents tensor product of two fuzzy graphs G_1 and G_2.

Theorem 11.2.1. *Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs where $i = 1, 2$. Then $NCI(G_1 \cup G_2) = \sum_m [(\vee\{e_{G_1}(m), e_{G_2}(m)\})\,(d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|)]$, where E_1 and E_2 are the edge sets of G_1 and G_2 and $|E_1 \cap E_2(m)|$ is the number of edges arising from the vertex m which lies in both G_1 and G_2.*

Proof. Consider $G_i = (\sigma_i, \mu_i)$, $i = 1, 2$. We prove this theorem by considering three cases. As the first case, we take $m \in V_1$ or $m \in V_2$, but not both. If $m \in V_1$ then $d_{G_1 \cup G_2}(m) = d_{G_1}(m)$ (since there is no new neighbor by construction.) $= d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|$ (since in this case $d_{G_2}(m) = |E_1 \cap E_2(m)| = 0$). Similar case arises when $m \in V_2$ also. Now consider the potential of the vertex in $G_1 \cup G_2$. For $m \in G_1$, $e_{G_1 \cup G_2}(m) = e_{G_1}(m)$ (since there is no new edge originating from m and there is no change in weight for the existing edges) $= \vee\{e_{G_1}(m), e_{G_2}(m)\}$ (since in this case $e_{G_2}(m) = 0$). Similarly for $m \in G_2$ also. As the second case, we take $m \in V_1 \cap V_2$, but no edge incident at m lies in $E_1 \cap E_2$. Here for $m \in V_1 \cap V_2$, $d_{G_1 \cup G_2}(m) = d_{G_1}(m) + d_{G_2}(m) = d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|$. While considering the potential of the vertex, $e_{G_1 \cup G_2}(m) = \vee\{e_{G_1}(m), e_{G_2}(m)\}$, since no edge incident at m lies in $E_1 \cap E_2$. As the third case, we take $m \in V_1 \cap V_2$, but some edges incident at m are in $E_1 \cap E_2$. Here for $m \in V_1 \cap V_2$, $d_{G_1 \cup G_2}(m) = d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|$. The potential of the vertex m is $e_{G_1 \cup G_2}(m) = \vee\{e_{G_1}(m), e_{G_2}(m)\}$, in this case, since the edges are taking the maximum weight and the maximum will be any of the $e_{G_i}(m)$, $i = 1, 2$. From the three cases,

$$NCI(G_1 \cup G_2) = \sum_m [(\vee\{e_{G_1}(m), e_{G_2}(m)\})(d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|)].$$

■

Theorem 11.2.2. *Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs with $|\sigma_i^*| = n_i$ where $i = 1, 2$. Assuming $V_1 \cap V_2 = \phi$ we have*

$$NCI(G_1 + G_2) = \sum_{m \in G_i, i \neq j} [(d_{G_i}(m) + n_j)(\vee_{p \in G_j}\{\sigma(m) \wedge \sigma(p)\})].$$

Proof. Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs with $|\sigma_i^*| = n_i$ where $i = 1, 2$. Suppose $m \in G_1$, then the neighborhood of m has all elements in G_2 in addition to its neighborhood in G_1 itself. Therefore $d_{G_1+G_2}(m) = d_{G_1}(m) + n_2$. Similarly if $m \in G_2$, $d_{G_1+G_2}(m) = d_{G_2}(m) + n_1$. Now we can check the potential of the vertex $m \in G_1$. Since $V_1 \cap V_2 = \phi$ there are two types of edges arising from m. One is those edges whose other endpoint is in G_1 and other is those edges whose other endpoint is in G_2. Edges of the first case have maximum connectedness $e_{G_1}(m)$. Edges of the second case have connectedness the minimum of the weight of its adjacent vertices. The maximum among them is greater than or equal to $e_{G_1}(m)$. Therefore $e_{G_1+G_2}(m) = \vee_{p \in G_2}\{\sigma(m) \wedge \sigma(p)\}$. Similarly if $m \in G_2$, therefore $e_{G_1+G_2}(m) = \vee_{p \in G_1}\{\sigma(m) \wedge \sigma(p)\}$.

Therefore

$$\begin{aligned} NCI(G_1 + G_2) &= \sum_{m \in G_1 + G_2} d_{G_1+G_2}(m) e_{G_1+G_2}(m) \\ &= \sum_{m \in G_1} [(d_{G_1}(m) + n_2)(\vee_{p \in G_2}\{\sigma(m) \wedge \sigma(p)\})] \end{aligned}$$

$$+\sum_{m\in G_2}[(d_{G_2}(m)+n_1)(\vee_{p\in G_1}\{\sigma(m)\wedge\sigma(p)\})]$$

$$=\sum_{m\in G_i, i\neq j}[(d_{G_i}(m)+n_j)(\vee_{p\in G_j}\{\sigma(m)\wedge\sigma(p)\})]. \qquad \blacksquare$$

Example 11.2.3. Consider the fuzzy graphs G_1 and G_2 with $\sigma_1^*=\{l,a,m\}$ and $\sigma_2^*=\{b,n\}$ where $\sigma(l)=0.9$, $\sigma(a)=0.4$, $\sigma(m)=1$, $\sigma(b)=0.8$, $\sigma(n)=0.7$, and $\mu(la)=0.3$, $\mu(am)=0.1$, $\mu(lm)=0.8$, $\mu(bn)=0.6$ (Fig. 11.9). After finding G_1+G_2 we calculate $NCI(G_1+G_2)=4\times 0.8+4\times 0.4+4\times 0.8+4\times 0.8+4\times 0.7=14$. Now using Theorem 11.2.2 we can find this without actually finding G_1+G_2, $NCI(G_1+G_2)=(2+2)\vee\{0.8,0.7\}+(2+2)\vee\{0.4,0.4\}+(2+2)\vee\{0.8,0.7\}+(1+3)\vee\{0.8,0.4,0.8\}+(1+3)\vee\{0.7,0.4,0.7\}=4\times 0.8+4\times 0.4+4\times 0.8+4\times 0.8+4\times 0.7=14$.

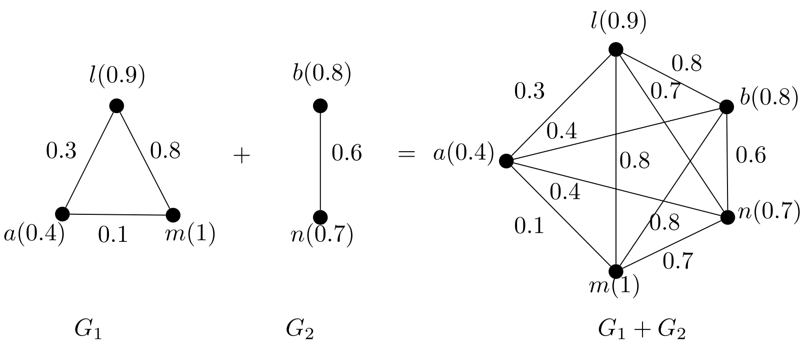

FIGURE 11.9

Join of two fuzzy graphs.

Theorem 11.2.4. *Let $G_i=(\sigma_i,\mu_i)$ be fuzzy graphs with $|\sigma_i^*|=n_i$ where $i=1,2$.*

(i) if $\sigma_1\leq\mu_2$, then

$$NCI(G_1[G_2])=\sum_{(m,p)\in V_1\times V_2}[n_2d_{G_1}(m)+d_{G_2}(p)]\sigma_1(m).$$

(ii) if $\sigma_1\geq\mu_2$, $\sigma_2\geq\mu_1$, then

$$NCI(G_1[G_2])=\sum_{(m,p)\in V_1\times V_2}[n_2d_{G_1}(m)+d_{G_2}(p)][\vee\{e_{G_1}(m),e_{G_2}(p)\}].$$

Proof. Let G_1 and G_2 be two fuzzy graphs. Then,

$$NCI(G_1[G_2])=\sum_{(m,p)\in V_1\times V_2}d_{G_1[G_2]}(m,p)e_{G_1[G_2]}(m,p).$$

We can calculate $d_{G_1[G_2]}(m, p)$ and $e_{G_1[G_2]}(m, p)$ separately. The neighborhood of (m, p) consists of three types of vertices. (1) $\{(x, y) : x = m, py \in E_2\}$, the cardinality of this set is $d_{G_2}(p)$. (2) $\{(x, y) : y = p, mx \in E_1\}$, the cardinality of this set is $d_{G_1}(m)$. (3) $\{(x, y) : y \neq p, mx \in E_1\}$, the cardinality of this set is $(n_2 - 1)d_{G_1}(m)$. Therefore $d_{G_1[G_2]}(m, p) = d_{G_2}(p) + d_{G_1}(m) + (n_2 - 1)d_{G_1}(m) = n_2 d_{G_1}(m) + d_{G_2}(p)$. Now we can look into the potential of the vertex (m, p).

$$
\begin{aligned}
&e_{G_1[G_2]}(m, p)\\
&= \vee_{(m,p)\in V_1\times V_2}\{\mu_{G_1[G_2]}((m, p)(x, y)) : (x, y) \in V_1 \times V_2\}\\
&= \vee_{(m,p)\in V_1\times V_2}\begin{cases} \sigma_1(m) \wedge \mu_2(py) & : \text{if} \quad x = m, py \in E_2\\ \sigma_2(p) \wedge \mu_1(mx) & : \text{if} \quad y = p, mx \in E_1\\ \sigma_2(p) \wedge \sigma_2(y) \wedge \mu_1(mx) & : \text{if} \quad y \neq p, mx \in E_1 \end{cases} \qquad (*)
\end{aligned}
$$

Now we can analyze the three parts of the theorem.

Part i: $\sigma_1 \leq \mu_2$. By Theorem 11.1.1 Eq. (*) becomes

$$
e_{G_1[G_2]}(m, p) = \vee_{(m,p)\in V_1\times V_2}\begin{cases} \sigma_1(m) & : \text{if} \quad x = m, py \in E_2\\ \mu_1(mx) & : \text{if} \quad y = p, mx \in E_1\\ \mu_1(mx) & : \text{if} \quad y \neq p, mx \in E_1 \end{cases} = \sigma_1(m).
$$

Therefore $NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} [n_2 d_{G_1}(m) + d_{G_2}(p)]\sigma_1(m)$.

Part ii: $\sigma_1 \geq \mu_2, \sigma_2 \geq \mu_1$. Here Eq. (*) becomes

$$
\begin{aligned}
e_{G_1[G_2]}(m, p) &= \vee_{(m,p)\in V_1\times V_2}\begin{cases} \mu_2(py) & : \text{if} \quad x = m, py \in E_2\\ \mu_1(mx) & : \text{if} \quad y = p, mx \in E_1\\ \mu_1(mx) & : \text{if} \quad y \neq p, mx \in E_1 \end{cases}\\
&= \vee\{e_{G_1}(m), e_{G_2}(p)\}.
\end{aligned}
$$

Therefore

$$
NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} [n_2 d_{G_1}(m) + d_{G_2}(p)][\vee\{e_{G_1}(m), e_{G_2}(p)\}]. \quad \blacksquare
$$

Corollary 11.2.5. *Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs with $|\sigma_i^*| = n_i$ where $i = 1, 2$.*

(i) if $\sigma_1 \leq \mu_2$, then

$$
NCI(G_1 \times G_2) = \sum_{(m,p)\in V_1\times V_2} [d_{G_1}(m) + d_{G_2}(p)]\sigma_1(m).
$$

(ii) if $\sigma_1 \geq \mu_2$, $\sigma_2 \geq \mu_1$, then

$$
NCI(G_1 \times G_2) = \sum_{(m,p)\in V_1\times V_2} [d_{G_1}(m) + d_{G_2}(p)][\vee\{e_{G_1}(m), e_{G_2}(p)\}].
$$

Proof. Since by construction, the Cartesian product of two fuzzy graphs differs from composition only by the set of edges $\{xy : y \neq p, mx \in E_1\}$. There is no change for $e_{G_1 \times G_2}(m, p)$ from $e_{G_1[G_2]}(m, p)$ which can be observed from Eq. $(*)$ in Theorem 11.2.4. While considering the neighborhood of the vertex (m, p), third type mentioned in the above proof is missing. Therefore $d_{G_1 \times G_2}(m, p) = d_{G_1}(m) + d_{G_2}(p)$. Hence if $\sigma_1 \leq \mu_2$, then

$$NCI(G_1 \times G_2) = \sum_{(m,p) \in V_1 \times V_2} [d_{G_1}(m) + d_{G_2}(p)]\sigma_1(m)$$

and if $\sigma_1 \geq \mu_2, \sigma_2 \geq \mu_1$, then

$$NCI(G_1 \times G_2) = \sum_{(m,p) \in V_1 \times V_2} [d_{G_1}(m) + d_{G_2}(p)][\vee\{e_{G_1}(m), e_{G_2}(p)\}]. \qquad \blacksquare$$

Theorem 11.2.6. *Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs with $|\sigma_i^*| = n_i$ where $i = 1, 2$. Then*

$$NCI(G_1 \otimes G_2) = \sum_{(m,p) \in V_1 \times V_2} (d(m)d(p))(\wedge\{e(m), e(p)\}).$$

Proof. Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs with $|\sigma_i^*| = n_i$ where $i = 1, 2$. First, we find $d_{G_1 \otimes G_2}(m, p)$ and then $e_{G_1 \otimes G_2}(m, p)$. Consider the vertex $(m, p) \in V_1 \times V_2$. In the vertex set of $V_1 \times V_2$ we can find n_2 number of vertices with same first coordinate. Among the n_2 vertices there exists $d(p)$ vertices which has a neighborhood with (m, p). And this case repeats $d(m)$ times. Therefore $d_{G_1 \otimes G_2}(m, p) = d(m)d(p)$. Now

$$\begin{aligned} e_{G_1 \otimes G_2}(m, p) &= \vee\{\mu_{G_1 \otimes G_2}((m, p)(x, y)); (x, y) \in V_1 \times V_2\} \\ &= \vee\{\mu_{G_1}(mx) \wedge \mu_{G_2}(py); mx \in E_1 \text{and } py \in E_2\} \\ &= \wedge\{[\vee\mu_{G_1}(mx), \vee\mu_{G_2}(py)]; mx \in E_1 \text{and } py \in E_2\} \\ &= \wedge\{e(m), e(p)\}. \end{aligned}$$

Therefore

$$NCI(G_1 \otimes G_2) = \sum_{(m,p) \in V_1 \times V_2} (d(m)d(p))(\wedge\{e(m), e(p)\}). \qquad \blacksquare$$

11.3 Algorithm

This section discusses an algorithm to find the NCI of a fuzzy graph.

Algorithm 11.3.1. Let $G = (\sigma, \mu)$ be a fuzzy graph with n vertices.

1. Construct the matrix $A = [a_{ij}]$ with $a_{ij} = \mu(m_i m_j)$.

2. Find the largest membership value in each row of the matrix. Let it be t_i.
3. Find the number of nonzero entries in each row of the matrix. Let it be s_i.
4. Then $NCI(G) = \sum_{i=1}^{n} t_i \times s_i$.

Illustration of Algorithm: Let $A = (\sigma, \mu)$ be a fuzzy graph in Fig. 11.10 with $\sigma^* = \{l, a, m, b, n, c, o, d\}$ such that $\mu(la) = 0.5$, $\mu(lm) = 0.3$, $\mu(am) = 0.4$, $\mu(mb) = 0.1$, $\mu(bn) = 0.2$, $\mu(nc) = 0.3$, $\mu(no) = 0.6$, $\mu(nd) = 0.6$, $\mu(co) = 0.7$, $\mu(od) = 0.5$.

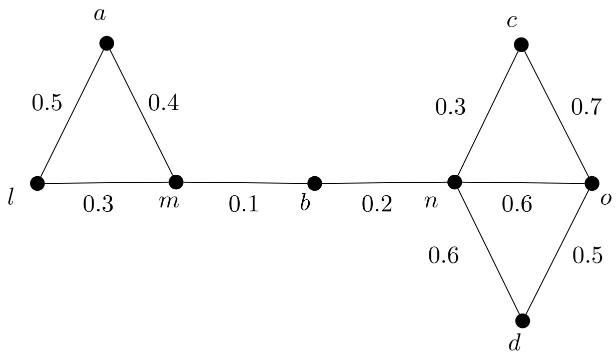

FIGURE 11.10

Illustration for Algorithm.

The matrix representation of the given fuzzy graph is

$$A = \begin{array}{c} \\ l \\ a \\ m \\ b \\ n \\ c \\ o \\ d \end{array} \begin{array}{c} \begin{array}{cccccccc} l & a & m & b & n & c & o & d \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.3 & 0.6 & 0.6 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.7 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.5 & 0 \end{pmatrix} \end{array}$$

Now NCI can be calculated by summing the product of highest value of each row and number of nonzero entries in each row. Here NCI, $NCI(G) = 0.5 \times 2 + 0.5 \times 2 + 0.4 \times 3 + 0.2 \times 2 + 0.6 \times 4 + 0.7 \times 2 + 0.7 \times 3 + 0.6 \times 2 = 10.7$.

11.4 Application

Human trafficking has always been widely studied since its impact on human race is huge. In 2017, Mathew and Mordeson discussed about this in [69]. Directed graph techniques was used by them to analyze the given data in Table 11.1. The ✓ in the data represents extremely small flow between regions and thus we neglect that further. The flow within a region is also not considered.

First of all, we construct the directed fuzzy graph S from this data. The vertices represent the regions and the directed edges represent the direction of the transition. Since the adjacency matrix of S is similar to the analyzed data, it is not mentioned again. Now by using the algorithm that we mentioned previously, we calculate NCI of S and it is 6.22. After several calculations we came to an assumption that there does not exist a twinning vertex set of cardinality one. Consider the vertex set $\{l, c\}$. Now we construct the adjacency matrix of $S \setminus \{l, c\}$.

$$S \setminus \{l, c\} = \begin{array}{c} \\ a \\ m \\ b \\ n \\ o \\ d \\ p \\ e \end{array} \begin{array}{c} \begin{array}{cccccccc} a & m & b & n & o & d & p & e \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.27 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 \\ 0.04 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0.06 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.07 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0.18 \\ 0.16 & 0 & 0 & 0 & 0 & 1.0 & 0 & 0.10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

After the computation using algorithm we get the NCI of $S \setminus \{l, c\}$ as 5.07. Next consider the vertex set $\{m, c\}$ and construct the adjacency matrix of $S \setminus \{m, c\}$.

$$S \setminus \{m, c\} = \begin{array}{c} \\ l \\ a \\ b \\ n \\ o \\ d \\ p \\ e \end{array} \begin{array}{c} \begin{array}{cccccccc} l & a & b & n & o & d & p & e \end{array} \\ \begin{pmatrix} 0 & 0.13 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.04 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0.06 \\ 0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.07 & 0.07 & 0 & 0.25 & 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0.18 \\ 0 & 0.16 & 0 & 0 & 0 & 1.0 & 0 & 0.10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Here also after the computation we get the NCI of $S \setminus \{m, c\}$ as 5.07. Therefore $\{l, c\}$ and $\{m, c\}$ are examples of twinning vertex sets of cardinality two. Similarly, we can see that $\{l, p\}$ and $\{m, d\}$ are also twinning vertex sets of cardinality two with NCI 2.58.

Table 11.1 Flows between different regions.

	WC Eur (l)	WS Eur (a)	C Eur & Bal (m)	E Eur & C Asia (b)	N Am & C Am & Car (n)	S Am (c)	E Asia & Pac (o)	S Asia (d)	S S Afr (p)	Mid East (e)
WC Eur (l)	0.62	0.13								
WS Eur (a)		0.16								
C Eur & Bal (m)		0.27	0.79		0.05					
E Eur & C Asia (b)	0.04	0.04	0.05	0.99						0.06
N Am & C Am & Car (n)	0.08	✓	✓		0.59	0.04				
S Am (c)		0.07			0.03	0.94				
E Asia & Pac(o)	0.07	0.07	✓		0.25	0.01	0.97			0.33
S Asia (d)		✓			0.07			0.96		0.18
S S Afr (p)		0.16						1.0		0.10
Mid East (e)										0.31

As this application gives the NCI of a human trafficking network, the value obtained is of real concern. We can subdivide the networks into smaller portions and compare the regions with different neighborhood connectivity. Also, this example gives the locality where we have to focus on the removal of the twinning vertex sets $\{l, p\}$ and $\{m, d\}$ provide a much lesser index. Controlling traffic through the locations l, p, m, and d can substantially reduce the traffic in the network (Fig. 11.11).

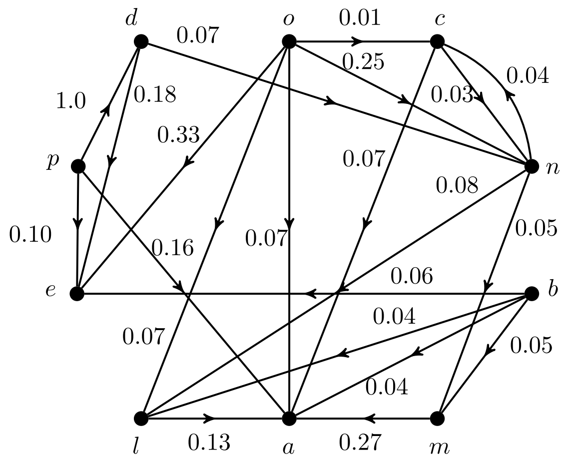

FIGURE 11.11

Directed fuzzy graph of the given data.

11.5 Exercises

1. Calculate NCI of $G = (\sigma, \mu)$ with $\sigma^* = \{l, a, m, b, n, c\}$; $\sigma(l) = 0.7$, $\sigma(a) = 0.8$, $\sigma(m) = 0.4$, $\sigma(b) = 0.9$, $\sigma(n) = 0.6$, $\sigma(c) = 0.5$, and $\mu(la) = 0.6$, $\mu(ln) = 0.3$, $\mu(lc) = 0.4$, $\mu(am) = 0.3$, $\mu(an) = 0.5$, $\mu(mn) = 0.4$, $\mu(bn) = 0.6$, $\mu(nc) = 0.5$.

2. Show that the NCI of the maximum spanning tree of a fuzzy tree G is always less than or equal to that G.

3. Construct a fuzzy graph with 5 vertices whose NCI is 6.

CHAPTER

Sigma index[⍟] 12

Among different graph indices, degree-based indices and distance based indices are generally more easy to handle. Among these, the sigma index, proposed by Gutman [43] related to vertex degrees, has gained more academic interest, particularly in mathematics and chemistry. Sigma index of fuzzy graphs is discussed in this chapter. The sigma index of well-known graph structures like cycles, saturated fuzzy cycles, paths, and stars are discussed. Relationships between sigma index and first and second Zagreb indices [44,45] are also discussed.

12.1 Sigma index of a fuzzy graph

This section discusses sigma index of a fuzzy graph, a graph index with several applications in different fields of science and technology including chemistry, information theory, and computer science.

Gutman et al. introduced the concept of the sigma index in their paper [43]. They discovered its similarity to the standard deviation in Statistics. Gutman also introduced the inverse sigma index of a graph. Later, several studies on sigma index were conducted by scientists from different fields. This section discusses the fuzzy sigma index and its various properties.

Definition 12.1.1. The sigma index, of a fuzzy graph $G=(\sigma,\mu)$ denoted by $\mathcal{S}(G)$ is defined as $\mathcal{S}(G)=\sum_{wm\in\mu^*}(d(w)-d(m))^2$, where the summation goes over all pairs of adjacent vertices w and m, and $d(w)$ and $d(m)$ are the fuzzy degrees of w and m, respectively.

Example 12.1.2. Consider $G=(\sigma,\mu)$ in Fig. 12.1 with $\sigma^*=\{w_1,w_2,w_3,w_4,w_5,w_6,w_7\}$ and $\mu(w_1w_2)=0.1$, $\mu(w_2w_3)=0.3$, $\mu(w_2w_5)=0.2$, $\mu(w_3w_4)=0.4$, $\mu(w_4w_5)=0.2$, $\mu(w_4w_6)=0.1$, $\mu(w_4w_7)=0.3$, $\mu(w_5w_6)=0.3$.

Given fuzzy graph has eight edges. For each edge, find the difference between fuzzy degree of their end vertices and then find its square. Consider the edge w_1w_2,

[⍟] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00020-9

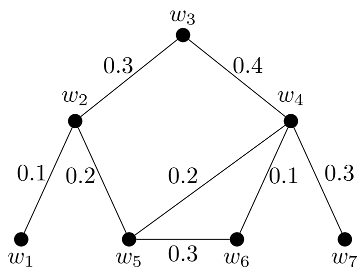

FIGURE 12.1

A fuzzy graph G with $\mathcal{S}(G) = 1.39$.

then $d(w_1) - d(w_2) = 0.5$. Now the square gives 0.25 as one of the summands. Similarly, we find the same for the rest of edges also. Tabulations are given in Table 12.1.1 and Table 12.1.2. From Table 12.1.1 and Table 12.1.2, we can find the sigma index

Table 12.1.1 Degree of vertices of G.

vertex	w_1	w_2	w_3	w_4	w_5	w_6	w_7
d(vertex)	0.1	0.6	0.7	1	0.7	0.4	0.3

Table 12.1.2 Calculation of sigma index.

edge(ab)	$d(a) - d(b)$	$[d(a) - d(b)]^2$
w_1w_2	-0.5	0.25
w_2w_3	-0.1	0.01
w_2w_5	-0.1	0.01
w_3w_4	-0.3	0.09
w_4w_5	0.3	0.09
w_4w_6	0.6	0.36
w_4w_7	0.7	0.49
w_5w_6	0.3	0.09
	$\mathcal{S}(G)$	1.39

of the given fuzzy graph as 1.39.

Unlike other indices, the sigma index of a partial fuzzy subgraph need not be less than or equal to that of the mother graph. Consider the following examples:

Example 12.1.3. The sigma index of a fuzzy graph can be lesser or greater or equal to the sigma index of its partial fuzzy subgraph. With suitable examples, it is shown below.

Case 1: Consider $H_1 = (\sigma_1, \nu_1)$, given in Fig. 12.2, partial fuzzy subgraph of $G = (\sigma, \mu)$ in Fig. 12.1 with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ and $\nu(w_1w_2) = 0.1$,

$\nu(w_2w_3) = 0.3$, $\nu(w_2w_5) = 0.2$, $\nu(w_3w_4) = 0.4$, $\nu(w_4w_5) = 0$, $\nu(w_4w_6) = 0.1$, $\nu(w_4w_7) = 0.3$, $\nu(w_5w_6) = 0.3$. Then sigma index of G is 1.39 and sigma index of H_1 is 0.7. i.e., $\mathcal{S}(H) < \mathcal{S}(G)$.

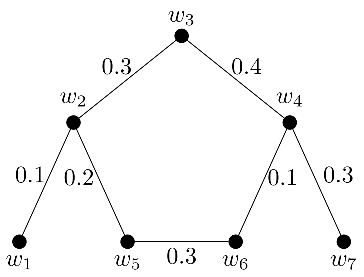

FIGURE 12.2

A partial fuzzy subgraph H_1 of G with $\mathcal{S}(H_1) = 0.7$.

Case 2: Consider $H_2 = (\sigma_2, \nu_2)$, given in Fig. 12.3, another partial fuzzy subgraph of $G = (\sigma, \mu)$ in Fig. 12.1 with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ and $\nu(w_1w_2) = 0.1$, $\nu(w_2w_3) = 0$, $\nu(w_2w_5) = 0.2$, $\nu(w_3w_4) = 0.4$, $\nu(w_4w_5) = 0.2$, $\nu(w_4w_6) = 0.1$, $\nu(w_4w_7) = 0.3$, $\nu(w_5w_6) = 0.3$. Then sigma index of G is 1.39 and sigma index of H_1 is 1.59. i.e., $\mathcal{S}(H) > \mathcal{S}(G)$.

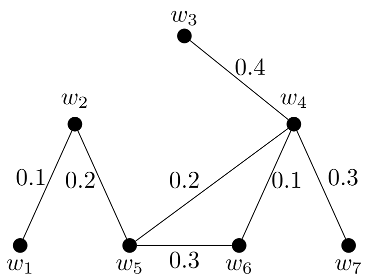

FIGURE 12.3

A partial fuzzy subgraph H_2 of G with $\mathcal{S}(H_2) = 1.59$.

Case 3: Consider $G = (\sigma, \mu)$ in Fig. 12.4 with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and $\mu(w_1w_2) = 0.4$, $\mu(w_2w_3) = 0.3$, $\mu(w_3w_1) = 0.1$, $\mu(w_3w_4) = 0.2$, $\mu(w_4w_5) = 0.2$, $\mu(w_5w_6) = 0.2$. Let $H = (\sigma, \nu)$ in Fig. 12.5 be a fuzzy subgraph of G with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and $\mu(w_1w_2) = 0.4$, $\mu(w_2w_3) = 0.3$, $\mu(w_3w_1) = 0.1$, $\mu(w_3w_4) = 0.2$, $\mu(w_4w_5) = 0.2$, $\mu(w_5w_6) = 0$. Then sigma index of G and H are 0.14. i.e., $\mathcal{S}(H) = \mathcal{S}(G)$.

The following theorem and corollaries discuss certain properties of cycles and fuzzy cycles. Since cycles are generalized concepts of fuzzy cycles, results true for cycles are also true for fuzzy cycles. Also whenever t or t_i are referenced in this article, their values fall between 0 and 1.

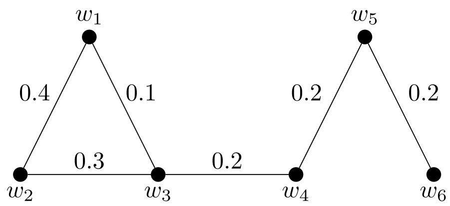

FIGURE 12.4

A fuzzy graph G with $\mathcal{S}(G) = 0.14$.

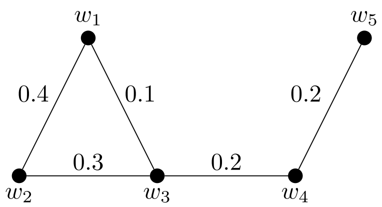

FIGURE 12.5

Partial fuzzy subgraph H of G with $\mathcal{S}(H) = 0.14$.

Theorem 12.1.4. *For a cycle C_n with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_i w_{i+1}$ having $\mu(e_i) = t_i > 0$, we have*

$$\mathcal{S}(C_n) = 2[\sum_{i=1}^{n} t_{i-1}^2] - 2[\sum_{i=1}^{n} t_{i-1}t_{i+1}],$$

where i is taken under modulo n.

Proof. Let C_n be a cycle as stated in the theorem. Consider an arbitrary vertex w_i. Fuzzy degree of w_i, $d(w_i) = t_{i-1} + t_i$. Therefore the sigma index is given as,

$$\begin{aligned}\mathcal{S}(C_n) &= \sum_{i=1}^{n}[d(w_i) - d(w_{i+1})]^2 = \sum_{i=1}^{n}[(t_{i-1} + t_i) - (t_i + t_{i+1})]^2 \\ &= \sum_{i=1}^{n}[t_{i-1} - t_{i+1}]^2 = \sum_{i=1}^{n}[t_{i-1}^2 - 2t_{i-1}t_{i+1} + t_{i+1}^2] \\ &= 2[\sum_{i=1}^{n} t_{i-1}^2] - 2[\sum_{i=1}^{n} t_{i-1}t_{i+1}].\end{aligned}$$

■

Next corollary shows an easy way to find the sigma index of certain large cycles. We can also write $2[\sum_{i=1}^{n} t_{i-1}^2] - 2[\sum_{i=1}^{n} t_{i-1}t_{i+1}]$ as $2[\sum_{i=1}^{n} t_{i-1}(t_{i-1} - t_{i+1})]$. This new formula is used in the upcoming results.

Corollary 12.1.5. *Let C_n be a cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_j = w_j w_{j+1}$ having $\mu(e_j) = t_j > 0$. Let C_{in}, $i \in \mathbb{N}$ be cycles with vertex set $\{w_1, w_2, \cdots, w_{in}\}$ and edge set $\{e_1, e_2, \cdots, e_{in}\}$ where $e_j = w_j w_{j+1}$ having $\mu(e_j) = t_j$, for $1 \le j \le n$, the same t_j mentioned in C_n and $\mu(e_k) = \mu(e_l)$, $k = l \mod n$. Then*

$$\mathcal{S}(C_{in}) = i \times \mathcal{S}(C_n).$$

Proof. From Theorem 12.1.4 we have sigma index of a cycle is $2[\sum_{j=1}^{n} t_{j-1}(t_{j-1} - t_{j+1})]$. Then

$$\begin{aligned}\mathcal{S}(C_{in}) &= 2[\sum_{j=1}^{in} t_{j-1}(t_{j-1} - t_{j+1})]\\ &= 2[\sum_{j=1}^{n} t_{j-1}(t_{j-1} - t_{j+1})] + 2[\sum_{j=n+1}^{2n} t_{j-1}(t_{j-1} - t_{j+1})] + \cdots\\ &\quad + 2[\sum_{j=[(i-1)n]+1}^{in} t_{j-1}(t_{j-1} - t_{j+1})]\\ &= i \times 2[\sum_{j=1}^{n} t_{j-1}(t_{j-1} - t_{j+1})],\end{aligned}$$

by condition given in the statement $= i \times \mathcal{S}(C_n)$. ■

The following corollary represents the saturated fuzzy cycle's sigma index.

Corollary 12.1.6. *Let C_n be a saturated fuzzy cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_j = w_j w_{j+1}$. Suppose that all its $\beta-$ edges have weight x and $\alpha-$ edges have weight $t_1, t_2, \cdots, t_{\frac{n}{2}}$. i.e., $\mu(e_{2i+1}) = x$, for $0 \le i \le \frac{n-2}{2}$ and $\mu(e_{2i}) = t_i$, for $1 \le i \le \frac{n}{2}$, then*

$$\mathcal{S}(C_n) = 2[\sum_{i=1}^{\frac{n}{2}} t_i^2] - 2[\sum_{i=1}^{\frac{n}{2}} t_i t_{i+1}],$$

where i is taken under modulo $\frac{n}{2}$.

Proof. Let C_n be a cycle as stated in the theorem. Fuzzy degree of $w_i's$ are given as, $d(w_{2i}) = d(w_{2i+1}) = t_i + x$ for $1 \le i \le \frac{n}{2}$. Therefore the sigma index is given as,

$$\begin{aligned}
\mathcal{S}(C_n) &= \sum_{i=1}^{n}[d(w_i) - d(w_{i+1})]^2 \\
&= \sum_{i=1}^{\frac{n}{2}}[d(w_{2i}) - d(w_{2i+1})]^2 + \sum_{i=1}^{\frac{n}{2}}[d(w_{2i+1}) - d(w_{2i+2})]^2 \\
&= \sum_{i=1}^{\frac{n}{2}}[(t_i + x) - (t_i + x)]^2 + \sum_{i=1}^{\frac{n}{2}}[(t_i + x) - (t_{i+1} + x)]^2 \\
&= 0 + \sum_{i=1}^{\frac{n}{2}}[t_i - t_{i+1}]^2 = \sum_{i=1}^{\frac{n}{2}}[t_i^2 - 2t_i t_{i+1} + t_{i+1}^2] \\
&= 2[\sum_{i=1}^{\frac{n}{2}} t_i^2] - 2[\sum_{i=1}^{\frac{n}{2}} t_i t_{i+1}].
\end{aligned}$$ ■

While considering saturated fuzzy cycles with every $\alpha-$ strong edge having weight s and every $\beta-$ strong edge having weight t, we can see that, their sigma index is zero. A generalized version of the previous argument is given in the following corollary.

Corollary 12.1.7. *Let C_n be a cycle with $|\sigma^*| = 2n$. If alternative edges share same weight, then $\mathcal{S}(C_n) = 0$.*

Corollary 12.1.8. *Let P be a path with $\sigma^* = \{w_1, w_2, \cdots, w_n\}$ and $\mu(w_i w_{i+1}) = t_i$, $1 \leq i \leq n-1$, $n > 3$. Then*

$$\mathcal{S}(P) = t_1^2 + 2[\sum_{i=2}^{n-2} t_i^2] + t_n^2 - 2[\sum_{i=1}^{n-3} t_i t_{i+2}].$$

Theorem 12.1.9. *Let C_n be a cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ with $e_j = w_j w_{j+1}$ having $\mu(e_j) = \mu(e_{j-1}) + x$, $j = 2$ to n and $x \in (0, 1)$, $\sigma(w_n) \in (0, 1]$. Then*

$$\mathcal{S}(C_n) = 2n(n-2)x^2.$$

Proof. Consider a cycle C_n as stated in the theorem. Suppose $\mu(e_1) = t$, then $\mu(e_2) = t + x$, $\mu(e_3) = t + 2x$, $\cdots, \mu(e_n) = t + (n-1)x$. Which gives $d(w_1) = 2t + (n-1)x$, $d(w_2) = 2t + x$, $\cdots, d(w_n) = 2t + (2n-3)x$. Then $\mathcal{S}(C_n) = \sum_{i=1}^{n}[d(w_i) - d(w_{i+1})]^2 = ((n-2)x)^2 + \underbrace{(2x)^2 + \cdots + (2x)^2}_{(n-2)times} + ((n-2)x)^2 = (n-2)(2x)^2 + 2((n-2)x)^2 = 2n(n-2)x^2$. ■

Next we determine the sigma index of a star.

Theorem 12.1.10. *Let $S = (\sigma, \mu)$ be a star, with vertex set $\{w_1, w_2, \cdots, w_{n+1}\}$, where w_{n+1} is the central vertex and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_i w_{n+1}$ with $\mu(e_i) = t_i$. Then*

$$\mathcal{S}(S) = \sum_{i=1}^{n}\left[(n-1)t_i^2\right] + 2(n-2)\Big[\sum_{i=1}^{n}\Big[\sum_{j=i+1}^{n} t_i t_j\Big]\Big].$$

Proof. Consider S as stated in the theorem. Let w_i be an arbitrary vertex different from the central vertex. Then the fuzzy degree of w_i is t_i. The fuzzy degree of the central vertex w_{n+1} is $\sum_{i=1}^{n} t_i$. Therefore the sigma index

$$\begin{aligned}
\mathcal{S}(S) &= \sum_{i=1}^{n}[d(w_{n+1}) - d(w_i)]^2\\
&= \sum_{i=1}^{n}\Big[(\sum_{j=1}^{n} t_j) - t_i\Big]^2\\
&= \sum_{i=1}^{n}\Big[(\sum_{j=1}^{n} t_j)^2 - 2t_i\Big[\sum_{j=1}^{n} t_j\Big] + t_i^2\Big]\\
&= n(\sum_{i=1}^{n} t_i)^2 + \sum_{i=1}^{n}[-2t_i(\sum_{j=1}^{n} t_j) + t_i^2]\\
&= n(\sum_{i=1}^{n} t_i)^2 - 2\Big[\sum_{i=1}^{n} t_i \sum_{j=1}^{n} t_j\Big] + \sum_{i=1}^{n} t_i^2\\
&= (n-2)\Big[\sum_{i=1}^{n} t_i\Big]^2 + \sum_{i=1}^{n} t_i^2\\
&= (n-2)\Big[\sum_{i=1}^{n} t_i^2\Big] + 2(n-2)\Big[\sum_{i=1}^{n}\Big[\sum_{j=i+1}^{n} t_i t_j\Big]\Big] + \sum_{i=1}^{n} t_i^2\\
&= (n-1)\Big[\sum_{i=1}^{n} t_i^2\Big] + 2(n-2)\Big[\sum_{i=1}^{n}\Big[\sum_{j=i+1}^{n} t_i t_j\Big]\Big].
\end{aligned}$$

■

Remark. Two isomorphic fuzzy graphs pose the same sigma index.

A fuzzy graph G and another fuzzy graph $G + wm$ formed by adding a pendant vertex to G have the same sigma index. $N(w)$ represents the neighborhood of the vertex w.

Theorem 12.1.11. *Let G be a fuzzy graph. Let a vertex w of G be joined to a new vertex $m \notin \sigma^*$ by a new pendent edge wm. Then*

$$\mathcal{S}(G+wm)=\mathcal{S}(G) \text{ if and only if } \frac{ks(s+2t)+t^2}{2s}=\sum_{i=1}^{k} d(w_i),$$

where k is the degree of w in G, t is the fuzzy degree of w in G, $d_G(w_i)$'s be the fuzzy degree of vertices incident at w and s be the weight of the edge wm.

Proof. Consider a fuzzy graph G. Let w be a vertex in σ^* with $\{w_1, w_2, \cdots, w_k\}$ as its adjacent vertices. Let $\mu(ww_i)$ be t_i. Then the degree of w is k and the fuzzy degree is $\sum_{i=1}^{k} t_i = t$. Then

$$\begin{aligned}\mathcal{S}(G)=&\sum_{p_i,p_j\notin N(N(w))}(d_G(p_i)-d_G(p_j))^2+\sum_{w_i,w_j\in N(N(w))}(d_G(w_i)-d_G(w_j))^2\\&+\sum_{w_i\in N(N(w)),y\notin N(N(w))}(d_G(w_i)-d_G(y))^2+\sum_{i=1}^{k}(d_G(w)-d_G(w_i))^2\end{aligned}$$

For convenience, let us denote

$$\begin{aligned}&\sum_{p_i,p_j\notin N(N(w))}(d_G(p_i)-d_G(p_j))^2+\sum_{w_i,w_j\in N(N(w))}(d_G(w_i)-d_G(w_j))^2\\&+\sum_{w_i\in N(N(w)),y\notin N(N(w))}(d_G(w_i)-d_G(y))^2\end{aligned}$$

as $\mathcal{S}(G\setminus w)$. Then,

$$\begin{aligned}\mathcal{S}(G)&=\mathcal{S}(G\setminus w)+\sum_{i=1}^{k}(t-d_G(w_i))^2\\&=\mathcal{S}(G\setminus w)+\sum_{i=1}^{k}t^2+\sum_{i=1}^{k}d_G(w_i)^2-2t\sum_{i=1}^{k}d_G(w_i)\\&=\mathcal{S}(G\setminus w)+kt^2+\sum_{i=1}^{k}d_G(w_i)^2-2t\sum_{i=1}^{k}d_G(w_i) \qquad \text{(i)}\end{aligned}$$

Now consider a vertex $m\notin\sigma^*$ such that $\mu(wm)=s$. Then

$$\mathcal{S}(G+wm)=\sum_{p_i,p_j\notin N(N(w))}(d_{G+wm}(p_i)-d_{G+wm}(p_j))^2$$

$$+\sum_{w_i,w_j\in N(N(w))}(d_{G+wm}(w_i)-d_{G+wm}(w_j))^2$$

$$+\sum_{\substack{w_i\in N(N(w))\\ y\notin N(N(w))}}(d_{G+wm}(w_i)-d_{G+wm}(y))^2$$

$$+\sum_{i=1}^{k}(d_{G+wm}(w)-d_{G+wm}(w_i))^2+(d_{G+wm}(w)-d_{G+wm}(m))^2$$

$$=\sum_{p_i,p_j\notin N(N(w))}(d_G(p_i)-d_G(p_j))^2+\sum_{w_i,w_j\in N(N(w))}(d_G(w_i)-d_G(w_j))^2$$

$$+\sum_{\substack{w_i\in N(N(w))\\ y\notin N(N(w))}}(d_G(w_i)-d_G(y))^2+\sum_{i=1}^{k}(d_{G+wm}(w)-d_{G+wm}(w_i))^2$$

$$+(d_{G+wm}(w)-d_{G+wm}(m))^2$$

$$=\mathcal{S}(G\setminus w)+\sum_{i=1}^{k}(t+s-d_G(w_i))^2+(t+s-s))^2$$

$$=\mathcal{S}(G\setminus w)+\sum_{i=1}^{k}t^2+\sum_{i=1}^{k}s^2+\sum_{i=1}^{k}d_G(w_i)^2+2\sum_{i=1}^{k}ts-2t\sum_{i=1}^{k}d_G(w_i)$$

$$-2s\sum_{i=1}^{k}d_G(w_i)+t^2$$

$$=\mathcal{S}(G\setminus w)+kt^2+ks^2+\sum_{i=1}^{k}d_G(w_i)^2+2kts-2t\sum_{i=1}^{k}d_G(w_i)$$

$$-2s\sum_{i=1}^{k}d_G(w_i)+t^2 \qquad \text{(ii)}$$

From Eqs. (i) and (ii), $\mathcal{S}(G+wm)-\mathcal{S}(G)=ks^2+2kts-2s\sum_{i=1}^{k}d_G(w_i)+t^2$.

Therefore $\mathcal{S}(G+wm)=\mathcal{S}(G)$ if and only if $ks^2+2kts-2s\sum_{i=1}^{k}d_G(w_i)+t^2=0$.

i.e.,

$$\mathcal{S}(G+wm)=\mathcal{S}(G)\ \text{ if and only if }\ \frac{ks(s+2t)+t^2}{2s}=\sum_{i=1}^{k}d_G(w_i) \qquad \text{(iii)}$$

■

A special case of the above theorem is discussed in the next corollary.

Corollary 12.1.12. *Let G be a fuzzy graph with a pendant vertex w. Suppose w_1 is a pendant vertex in $G\backslash w$ and it is the supporting vertex of w in G with $d_G(w_1) = 2d_G(w)$, then G and $G\backslash w$ share same sigma index. Also, any fuzzy graph obtained by adjoining a path of length n to the pendant vertex w, with each edge having weight the same as that of the fuzzy degree of the pendant vertex w, share the same sigma index.*

Proof. Let G be a fuzzy graph and G' be the fuzzy graph obtained by adjoining a path of length 1 as stated in the corollary. Let $d_G(w)$ be s. By Theorem 12.1.11, $\mathcal{S}(G) = \mathcal{S}(G')$ if and only if $\frac{ks(s+2t)+t^2}{2s} = \sum_{i=1}^{k} d_G(w_i)$. Here $k = 1$, $d_G(w) = t = s$, $d_G(w_1) = 2s$. Which gives the LHS and RHS of Eq. (iii) as $2s$. Therefore $\mathcal{S}(G) = \mathcal{S}(G')$. Similarly adjoining a path of length n is also proved. ■

The coalescence of graphs where discussed by Brigham, Chinn, and Dutton in [18]. Now coalescence of two fuzzy graphs is discussed.

Definition 12.1.13. The **coalescence of fuzzy graphs** $G_1 = (W_1, \sigma_1, \mu_1)$ and $G_2 = (W_2, \sigma_2, \mu_2)$ via w_{G_1} and w_{G_2} is the graph obtained from G_1 and G_2 by identifying $w_{G_1} \in \sigma^*(G_1)$ and $w_{G_2} \in \sigma^*(G_2)$ in a vertex labeled w, denoted by $(G_1.G_2)(w_{G_1}, w_{G_2} : w)(\sigma_1.\sigma_2, \mu_1.\mu_2)$ with

$$\sigma_1.\sigma_2(m) = \begin{cases} \sigma_1(m) & \text{if} \quad m \in W_1, m \neq w \\ \sigma_2(m) & \text{if} \quad m \in W_2, m \neq w \\ \sigma_1(w_{G_1}) \vee \sigma_2(w_{G_2}) & \text{if} \quad m = w \end{cases},$$

$$\mu_1.\mu_2(m_1m_2) = \begin{cases} \mu_1(m_1m_2) & \text{if} \quad m_1m_2 \in E(G_1) \\ \mu_2(m_1m_2) & \text{if} \quad m_1m_2 \in E(G_2) \end{cases},$$

where, $\sigma_1 \circ \sigma_2$ is a fuzzy subset of $W = (W_1 \setminus \{w_{G_1}\}) \cup (W_2 \setminus \{w_{G_2}\}) \cup \{w\}$ and $\mu_1 \circ \mu_2$ is a fuzzy subset of $E = E(G_1) \cup E(G_2)$.

The sigma index of coalescence of two fuzzy graphs is discussed in the following theorem.

Theorem 12.1.14. *Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1| = n_1$ and $|\sigma_2| = n_2$. Let $\{s_1, s_2, \cdots, s_{n_1}\}$ be the vertex set of G_1 and $\{r_1, r_2, \cdots, r_{n_2}\}$ be the vertex set of G_2. Let s_i and r_j be arbitrary vertices from G_1 and G_2. Then the sigma index of fuzzy graph coalescence of G_1 and G_2 at s_i and r_j,*

$$\begin{aligned}
&\mathcal{S}(G_1.G_2)(s_i, r_j : w_*) \\
&\quad = \mathcal{S}(G_1) + \mathcal{S}(G_2) + deg_{G_1}(w_*)d_{G_2}(w_*)(2d_{G_1}(w_*) + d_{G_2}(w_*)) \\
&\qquad + deg_{G_2}(w_*)d_{G_1}(w_*))(d_{G_1}(w_*) + 2d_{G_2}(w_*)) \\
&\qquad - 2d_{G_1}(w_*) \sum_{w_i w_* \in G_1} d_{G_2}(w_i) - 2d_{G_2}(w_*) \sum_{w_i w_* \in G_2} d_{G_1}(w_i).
\end{aligned}$$

Proof. Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1| = n_1$ and $|\sigma_2| = n_2$, with vertex set $\{s_1, s_2, \cdots, s_{n_1}\}$ and $\{r_1, r_2, \cdots, r_{n_2}\}$ respectively. Let s_i and r_j be two arbitrary vertices from G_1 and G_2 respectively. Let $(G_1.G_2)(s_i, r_j : w_*)$ with the vertex set $\{w_1, w_2, \cdots, w_{n_1+n_2-1}\}$ be the fuzzy graph coalescence of G_1 and G_2 at vertices s_i and r_j formed by identifying these vertices to a new vertex called w_*. Then the sigma index is,

$$\mathcal{S}(G_1.G_2)(s_i, r_j : w_*) = \sum_{w_i w_j \in \mu^*} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_j)]^2$$

The equation can be executed by categorizing the vertices into two distinct cases based on their characteristics.

Case 1: Consider those edges which are adjacent to w_*.

$$\begin{aligned}\sum_{w_i w_* \in \mu^*} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_*)]^2 &= \sum_{w_i w_* \in \mu^*_{G_1}} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_*)]^2 \\ &\quad + \sum_{w_i w_* \in \mu^*_{G_2}} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_*)]^2 \\ &= \sum_{w_i w_* \in \mu^*_{G_1}} [d_{G_1}(w_i) - d_{G_1}(w_*) - d_{G_2}(w_*)]^2 \\ &\quad + \sum_{w_i w_* \in \mu^*_{G_2}} [d_{G_2}(w_i) - d_{G_1}(w_*) - d_{G_2}(w_*)]^2\end{aligned}$$

Case 2: Consider those edges which are not adjacent to w_*.

$$\begin{aligned}\sum_{w_i w_j \in \mu^*} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_j)]^2 &= \sum_{w_i w_j \in \mu^*_{G_1}} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_j)]^2 \\ &\quad + \sum_{w_i w_j \in \mu^*_{G_2}} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_j)]^2\end{aligned}$$

From cases (1) and (2) it can be concluded that

$$\begin{aligned}&\mathcal{S}(G_1.G_2)(s_i, r_j : w_*) \\ &\quad = \sum_{w_i w_* \in \mu^*_{G_1}} [d_{G_1}(w_i) - d_{G_1}(w_*) - d_{G_2}(w_*)]^2 \\ &\qquad + \sum_{w_i w_* \in \mu^*_{G_2}} [d_{G_2}(w_i) - d_{G_1}(w_*) - d_{G_2}(w_*)]^2 \\ &\qquad + \sum_{w_i w_j \in \mu^*_{G_1}} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_j)]^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{w_i w_j \in \mu^*_{G_2}} [d_{G_1.G_2}(w_i) - d_{G_1.G_2}(w_j)]^2 \\
&= \mathcal{S}(G_1) + \mathcal{S}(G_2) \\
&+ \sum_{w_i w_* \in \mu^*_{G_1}} [(d_{G_2}(w_*))^2 - 2d_{G_1}(w_i)d_{G_2}(w_*) \\
&+ 2d_{G_1}(w_*)d_{G_2}(w_*)] + \sum_{w_i w_* \in \mu^*_{G_2}} [(d_{G_1}(w_*))^2 \\
&- 2d_{G_2}(w_i)d_{G_1}(w_*) + 2d_{G_1}(w_*)d_{G_2}(w_*)] \\
&= deg_{G_1}(w_*)(d_{G_2}(w_*))^2 + deg_{G_2}(w_*)(d_{G_1}(w_*))^2 \\
&+ 2deg_{G_1}(w_*)d_{G_1}(w_*)d_{G_2}(w_*) \\
&+ 2deg_{G_2}(w_*)d_{G_1}(w_*)d_{G_2}(w_*) \\
&- 2d_{G_1}(w_*) \sum_{w_i w_* \in G_1} d_{G_2}(w_i) - 2d_{G_2}(w_*) \sum_{w_i w_* \in G_2} d_{G_1}(w_i) \\
&= \mathcal{S}(G_1) + \mathcal{S}(G_2) + deg_{G_1}(w_*)d_{G_2}(w_*)(2d_{G_1}(w_*) \\
&+ d_{G_2}(w_*)) + deg_{G_2}(w_*)d_{G_1}(w_*))(d_{G_1}(w_*) + 2d_{G_2}(w_*)) \\
&- 2d_{G_1}(w_*) \sum_{w_i w_* \in G_1} d_{G_2}(w_i) \\
&- 2d_{G_2}(w_*) \sum_{w_i w_* \in G_2} d_{G_1}(w_i)
\end{aligned}
$$

■

Coalescence of odd and even cycles with an arbitrary fuzzy graph is discussed below.

Theorem 12.1.15. *Let G be a fuzzy graph and C_{2n} be a cycle of order $2n, n > 1$ with vertex set $\{w_1, w_2, \cdots, w_{2n}\}$ and edge set $\{e_1, e_2, \cdots, e_{2n}\}$ where $e_i = w_i w_{i+1}$ and $\mu(e_{2i}) = t$, for $1 \le i \le n$, $\mu(e_{2i+1}) = r$, for $1 \le i \le n$. Let $(G.C_{2n})(m_1, w : s)$ be the coalescence of G and C_{2n}, where m_1 is an arbitrary vertex of G. Then*

$$\mathcal{S}((G.C_{2n})(m_1, w : s)) = \mathcal{S}((G.C_{2p})(m_1, w : s)), n, p > 1.$$

Proof. Let G be an arbitrary fuzzy graph. C_{2n} a cycle of order $2n, n > 1$ and C_{2p} a cycle of order $2p$ as stated in the theorem. Let $m_1 \in G$ and an arbitrary vertex $w_{C_{2n}}$ of C_{2n} be the vertices chosen for coalescence of G and C_{2n}. We know $d_{C_{2n}}(w_i) = t + r$ for all vertices in C_{2n}. By definition, all vertices of C_{2n} except $w_{C_{2n}}$ have the same degree as in C_{2n} in $(G.C_{2n})(m_1, w : s)$. Similarly, all vertices of G except m_1 have the same degree as in G in $(G.C_{2n})(m_1, w : s)$. Then

$$
\begin{aligned}
&\mathcal{S}((G.C_{2n})(m_1, w : s)) \\
&= \sum_{\substack{m_i m_j \in \mu^*(G) \\ m_i, m_j \neq s}} [d(m_i) - d(m_j)]^2 + \sum_{x_i \in N_G(s)} [d(x_i) - d(s)]^2
\end{aligned}
$$

$$+ \sum_{y_i \in N_{C_{2n}}(s)} [d(y_i) - d(s)]^2 + \sum_{\substack{i=1 \\ w_i, w_{i+1} \neq s}}^{2n} [d(w_i) - d(w_{i+1})]^2$$

$$= \sum_{\substack{m_i m_j \in \mu^*(G) \\ m_i, m_j \neq s}} [d(m_i) - d(m_j)]^2 + \sum_{x_i \in N_G(s)} [d(x_i) - d(s)]^2$$

$$+ \sum_{y_i \in N_{C_{2n}}(s)} [d(y_i) - d(s)]^2 \quad \text{(a)}$$

Since all summands in the last term are zero.

When the sigma index of $(G.C_{2p})(m_1, w : s)$ is calculated, we can see similar arguments as above. While considering equation (a) in case of $(G.C_{2t})(m_1, w : s)$, we can see that, there may be more or less number of summands in the last term. Since all the summands in the last term are zero. We get that $\mathcal{S}((G.C_{2n})(m_1, w : s)) = \mathcal{S}((G.C_{2p})(m_1, w : s))$. ■

The proof of Theorem 12.1.16 is similar as that Theorem 12.1.15.

Theorem 12.1.16. *Let G be a fuzzy graph and C_{2n+1} be a cycle of order $2n+1$ with vertex set $\{w_1, w_2, \cdots, w_{2n+1}\}$ and edge set $\{e_1, e_2, \cdots, e_{2n+1}\}$ where $e_i = w_i w_{i+1}$ having $\mu(e_{2i}) = t$, for $1 \leq i \leq n$, $\mu(e_{2i+1}) = m$, for $1 \leq i \leq n$, and $\mu(e_{2n+1}) = x$. Let $(G.C_{2n+1})(p_1, w : s)$ be the coalescence of G and C_{2n+1}, where p_1 is an arbitrary vertex of G. Then*

1. $\mathcal{S}((G.C_{2n+1})(p_1, w_1 : s)) = \mathcal{S}((G.C_{2t+1})(p_1, w_1 : s))$ $n, t \in \mathbb{N}$.
2. $\mathcal{S}((G.C_{2n+1})(p_1, w_{2n+1} : s)) = \mathcal{S}((G.C_{2t+1})(p_1, w_{2t+1} : s))$ $n, t \in \mathbb{N}$.

Example 12.1.17. Consider the fuzzy graphs $G = (\sigma_1, \mu_1)$, $C_3 = (\sigma_2, \mu_2)$ and $C_5 = (\sigma_3, \mu_3)$ given in Figs. 12.6 and 12.7 with $\sigma_1^* = \{m_1, m_2, m_3\}$, $\sigma_2^* = \{w_1, w_2, w_3\}$,

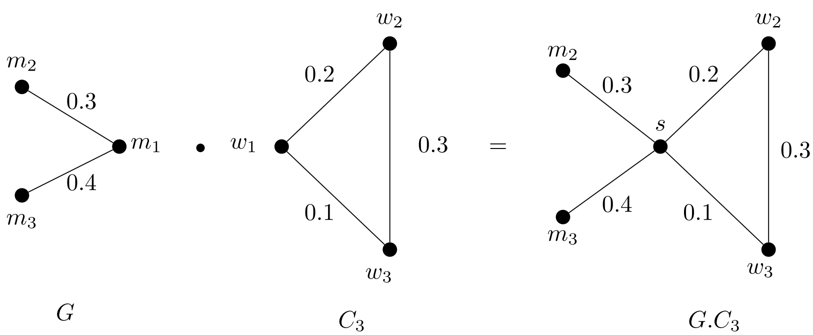

FIGURE 12.6

Fuzzy graphs G and C_3 with $\mathcal{S}((G.C_3)) = 1.47$.

and $\sigma_3^* = \{w_1, w_2, w_3, w_4, w_5\}$ where $\mu_1(m_1m_2) = 0.3$, $\mu_1(m_1m_3) = 0.4$, $\mu_2(w_1w_2) = 0.2$, $\mu_2(w_1w_3) = 0.1$, $\mu_2(w_2w_3) = 0.3$, $\mu_3(w_1w_2) = 0.2$, $\mu_3(w_1w_5) = 0.1$, $\mu_3(w_2w_3) = 0.3$, $\mu_3(w_3w_4) = 0.2$ and $\mu_3(w_4w_5) = 0.3$. After finding coalescence of G and C_3, which is shown in Fig. 12.6, we get $\mathcal{S}((G.C_3)(m_1, w_1 : s))$ as 1.47. Also after finding coalescence of G and C_5, which is shown in Fig. 12.7, we get $\mathcal{S}((G.C_5)(m_1, w_1 : s))$ as 1.47. Which gives $\mathcal{S}((G.C_3)(m_1, w_1 : s)) = \mathcal{S}((G.C_5)(m_1, w_1 : s))$.

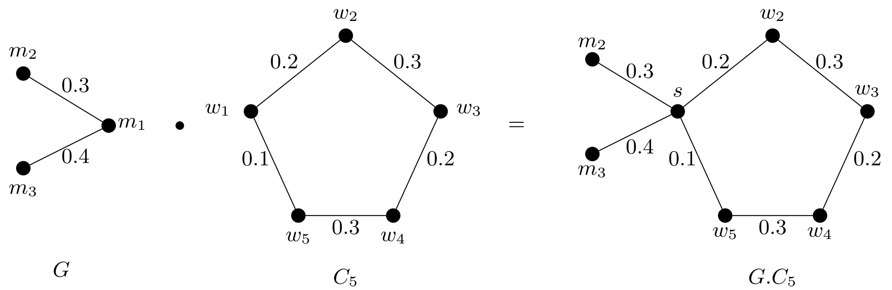

FIGURE 12.7

Fuzzy graphs G and C_5 with $\mathcal{S}((G.C_5)) = 1.47$.

Using some preset constraints and a given sigma index value, the next theorem describes how to design fuzzy graphs.

Theorem 12.1.18. *For a given $x \in \mathbb{R}^+$ there exists a fuzzy graph $G = (\sigma, \mu)$ of sigma index x.*

Proof. The theorem is proved in two parts. In the first part existence of such a fuzzy graph in the interval (0, 1] is proved, and then in the second part, it is extended to the positive real line. Let z be an arbitrary real number in the interval (0, 1]. Now choose e such that $e = \frac{z}{4}^{\frac{1}{2}}$. Now construct a cycle of length 4 as mentioned in Theorem 12.1.4 with $\mu(e_1) = x$, $\mu(e_2) = y$, $\mu(e_3) = x + e$ or $x - e$ and $\mu(e_4) = y + e$ or $y - e$ where x and y satisfies the following properties (1) $0 < x \leq 1$, (2) $0 < y \leq 1$, (3) $0 < x \pm e \leq 1$, (4) $0 < y \pm e \leq 1$. Now let us check the sigma index of the constructed graph. Let e_ie_j be the vertex where e_i and e_j incident. Therefore $d(e_1e_2) = x + y$, $d(e_2e_3) = x + y \pm e$, $d(e_3e_4) = x + y \pm e \pm e$ and $d(e_4e_1) = x + y \pm e$. Therefore the sigma index, $\mathcal{S}(G) = (\pm e)^2 + (\pm e)^2 + (\pm e)^2 + (\pm e)^2 = 4(\pm e)^2 = z$. Now using Corollary 12.1.5 we can extend this to the whole positive real axis. ∎

Suppose a fuzzy graph of sigma index 9 is to be made. Then construct a fuzzy graph with sigma index of 0.9. For that use the construction method mentioned in the above theorem. Now by using Corollary 12.1.5 construct a cycle of length $4 \times 10 = 40$. Which gives a fuzzy graph of sigma index $0.9 \times 10 = 9$.

The remaining portion of the section compares the sigma index to the first Zagreb index and the second Zagreb index.

Theorem 12.1.19. *Let G be a fuzzy graph, $\mathcal{S}(G)$ be its sigma index and $\mathcal{M}_2(G)$ be its second Zagreb index. Then*

$$\mathcal{S}(G) + 2\mathcal{M}_2(G) \leq \sum_{w\in\sigma^*} deg(w)d^2(w),$$

where $deg(w)$ is the degree of a vertex in a graph and $d(w)$ is the fuzzy degree of a vertex in fuzzy graph.

Proof. Consider a fuzzy graph G. Then

$$\begin{aligned}
&\mathcal{S}(G) + 2\mathcal{M}_2(G)\\
&\quad = \sum_{wm\in\mu^*} (d(w) - d(m))^2 + 2\sum_{wm\in\mu^*} \sigma(w)\sigma(m)d(w)d(m)\\
&\quad = \sum_{wm\in\mu^*} [d^2(w) + d^2(m) - 2d(w)d(m)] + 2\sum_{wm\in\mu^*} \sigma(w)\sigma(m)d(w)d(m)\\
&\quad = \sum_{wm\in\mu^*} [d^2(w) + d^2(m)] - 2\sum_{wm\in\mu^*} d(w)d(m)\\
&\qquad + 2\sum_{wm\in\mu^*} \sigma(w)\sigma(m)d(w)d(m)\\
&\quad \leq \sum_{wm\in\mu^*} [d^2(w) + d^2(m)] - 2\sum_{wm\in\mu^*} d(w)d(m) + 2\sum_{wm\in\mu^*} d(w)d(m)\\
&\quad = \sum_{wm\in\mu^*} [d^2(w) + d^2(m)]\\
&\quad = \sum_{w\in\sigma^*} deg(w)d^2(w).
\end{aligned}$$

■

Corollary 12.1.20. *Let G be a fuzzy graph with $\sigma(w) = 1$ for all $w \in \sigma^*$. Let $\mathcal{S}(G)$ be its sigma index, $\mathcal{M}_1(G)$ be its first Zagreb index and $\mathcal{M}_2(G)$ be its second Zagreb index. Then*

$$\mathcal{M}_1(G) \leq \mathcal{S}(G) + 2\mathcal{M}_2(G) \leq \sum_{w\in\sigma^*} deg(w)d^2(w).$$

12.2 Average sigma index of a fuzzy graph

In 2018, when Gutman et al., introduced the concept of the sigma index they mentioned that this graph invariant, denoted by σ, may be in resemblance with the standard deviation in statistics.

Table 12.2.1 Degree of vertices of G.

vertex	w_1	w_2	w_3	w_4	w_5	w_6	w_7
d(vertex)	0.1	0.6	0.7	1	0.7	0.4	0.3
d_A (vertex)	0.1	0.2	0.35	0.25	0.24	0.2	0.3

Table 12.2.2 Calculation of average sigma index.

edge(ab)	$d_A(a) - d_A(b)$	$[d_A(a) - d_A(b)]^2$
w_1w_2	−0.1	0.01
w_2w_3	−0.15	0.0225
w_2w_5	−0.04	0.0016
w_3w_4	0.1	0.01
w_4w_5	0.01	0.0001
w_4w_6	0.05	0.0025
w_4w_7	−0.05	0.0025
w_5w_6	0.04	0.0016
	$\mathcal{S}(G)$	0.0508

Definition 12.2.1. The average sigma index, of a fuzzy graph $G = (\sigma, \mu)$ denoted by $\mathcal{S}_{\mathcal{A}}(G)$ is defined as $\mathcal{S}_{\mathcal{A}}(G) = \sum_{wm \in \mu^*} (d_A(w) - d_A(m))^2$, where the summation goes over all pairs of adjacent vertices w and m, and $d_A(w) = \frac{d(w)}{deg(w)}$ and $d_A(m) = \frac{d(m)}{deg(m)}$ are the average degrees of w and m respectively.

Example 12.2.2. Consider the fuzzy graph given in Example 12.1.2. Here the fuzzy graph has eight edges. For each edge, we have to find the difference between the average degree of their end vertices and then square it. Consider edge w_1w_2, then $d_A(w_1) - d_A(w_2) = -0.1$. Now taking its square gives 0.01 as one of the summands. Similarly, we find for the rest of the edges. Tabulations are given below.

From Table 12.2.1 and Table 12.2.2, we can see that the average sigma index of the given fuzzy graph is 0.0508.

The average sigma index of cycles is studied below.

Theorem 12.2.3. *For a cycle C_n with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_iw_{i+1}$ having $\mu(e_i) = t_i > 0$, we have $\mathcal{S}_{\mathcal{A}}(C_n) = \frac{1}{2}[\sum_{i=1}^{n} t_i^2 - \sum_{i=1}^{n} t_it_{i+2}]$, where i is taken under modulo n.*

Proof. Let C_n be a cycle as stated in the theorem. Consider an arbitrary vertex w_i. Average degree of w_i, $d_A(w_i) = \frac{t_{i-1}+t_i}{2}$. Therefore the average sigma index is given as,

$$\mathcal{S}_{\mathcal{A}}(C_n) = \sum_{i=1}^{n}[d_A(w_i) - d_A(w_{i+1})]^2 = \sum_{i=1}^{n}\left[\frac{t_{i-1}+t_i}{2} - \frac{(t_i+t_{i+1})}{2}\right]^2$$
$$= \frac{1}{4}\sum_{i=1}^{n}[t_{i-1} - t_{i+1}]^2 = \frac{1}{2}[\sum_{i=1}^{n} t_i^2 - \sum_{i=1}^{n} t_i t_{i+2}],$$

by Theorem 12.1.4. ■

By Theorem 12.2.3 and Corollaries 12.1.5, 12.1.6, 12.1.7, 12.1.8 the following corollaries can be proved.

Corollary 12.2.4. *Let C_n be a cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_j = w_j w_{j+1}$ having $\mu(e_j) = t_j > 0$. Let C_{in}, $i \in \mathbb{N}$ be cycles with vertex set $\{w_1, w_2, \cdots, w_{in}\}$ and edge set $\{e_1, e_2, \cdots, e_{in}\}$ where $e_j = w_j w_{j+1}$ having $\mu(e_j) = t_j$, for $1 \le j \le n$, the same t_j mentioned in C_n and $\mu(e_k) = \mu(e_l)$, $k = l \mod n$. Then*

$$\mathcal{S}_{\mathcal{A}}(C_{in}) = i \times \mathcal{S}_{\mathcal{A}}(C_n).$$

Corollary 12.2.5. *Let C_n be a saturated fuzzy cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_j = w_j w_{j+1}$. Suppose that all its $\beta-$ edges have weight x and $\alpha-$ edges have weight $t_1, t_2, \cdots, t_{\frac{n}{2}}$. i.e., $\mu(e_{2i} + 1) = x$, for $0 \le i \le \frac{n-2}{2}$ and $\mu(e_{2i}) = t_i$, for $1 \le i \le \frac{n}{2}$, then*

$$\mathcal{S}_{\mathcal{A}}(C_n) = \frac{1}{4}\left[\sum_{i=1}^{\frac{n}{2}} t_i^2 - \sum_{i=1}^{\frac{n}{2}} t_i t_{i+1}\right],$$

where i is taken under modulo $\frac{n}{2}$.

Corollary 12.2.6. *Let C_n be a cycle with $|\sigma^*| = 2n$. If alternative edges share same weight, then $\mathcal{S}_{\mathcal{A}}(C_n) = 0$.*

Corollary 12.2.7. *Let P be a path with $\sigma^* = \{w_1, w_2, \cdots, w_n\}$ and $\mu(w_i w_{i+1}) = t_i$, $1 \le i \le n-1$, $n > 3$. Then*

$$\mathcal{S}_{\mathcal{A}}(P) = \frac{1}{2}\left[[\sum_{i=1}^{n} t_i^2] - [\sum_{i=1}^{n-3} t_i t_{i+2}] - t_1 t_2 - t_{n-2} t_{n-1}\right].$$

Theorem 12.2.8. *Let C_n be a cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ with $e_j = w_j w_{j+1}$ having $\mu(e_j) = \mu(e_{j-1}) + x$, $j = 2$ to n and $x \in (0, 1)$, $w_n \in (0, 1]$. Then*

$$\mathcal{S}_{\mathcal{A}}(C_n) = n(n-2)x^2.$$

Finding average sigma index of a star, is similar to the calculation of standard deviation.

Theorem 12.2.9. *Let $S=(\sigma,\mu)$ be a star, with vertex set $\{w_1, w_2, \cdots, w_{n+1}\}$, where w_{n+1} is the central vertex and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_i w_{n+1}$ with $\mu(e_i) = t_i$. Then*

$$\mathcal{S}_{\mathcal{A}}(S) = (1-\frac{1}{n})[\sum_{i=1}^{n} t_i^2] - \frac{2}{n}\Big[\sum_{i=1}^{n}[\sum_{j=i+1}^{n} t_i t_j]\Big].$$

Proof. Consider S as stated in the theorem. Let w_i be an arbitrary vertex different from the central vertex. Then the fuzzy degree of w_i is t_i. Which is the same as the average degree of $w_i's$. The average degree of the central vertex w_{n+1} is $\frac{\sum_{i=1}^{n} t_i}{n}$. Therefore the average sigma index

$$\begin{aligned}
\mathcal{S}_{\mathcal{A}}(S) &= \sum_{i=1}^{n}[d_A(w_{n+1}) - d_A(w_i)]^2 \\
&= \sum_{i=1}^{n}\left[\frac{1}{n}\left(\sum_{j=1}^{n} t_j\right) - t_i\right]^2 \\
&= \sum_{i=1}^{n}\left[\left[\frac{1}{n^2}(\sum_{j=1}^{n} t_j)^2\right] - 2t_i\left[\frac{1}{n}\sum_{j=1}^{n} t_j\right] + t_i^2\right] \\
&= \frac{1}{n}(\sum_{i=1}^{n} t_i)^2 + \sum_{i=1}^{n}\left[\frac{-2t_i}{n}(\sum_{j=1}^{n} t_j) + t_i^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} t_i\right]^2 - \frac{2}{n}\left[\sum_{i=1}^{n} t_i \sum_{j=1}^{n} t_j\right] + \sum_{i=1}^{n} t_i^2 \\
&= -\frac{1}{n}\Big[\sum_{i=1}^{n} t_i\Big]^2 + \sum_{i=1}^{n} t_i^2 \\
&= -\frac{1}{n}\Big[\sum_{i=1}^{n} t_i^2\Big] - \frac{2}{n}\Big[\sum_{i=1}^{n}[\sum_{j=i+1}^{n} t_i t_j]\Big] + \sum_{i=1}^{n} t_i^2 \\
&= (1-\frac{1}{n})[\sum_{i=1}^{n} t_i^2] - \frac{2}{n}\Big[\sum_{i=1}^{n}[\sum_{j=i+1}^{n} t_i t_j]\Big].
\end{aligned}$$

■

12.3 Algorithm

The algorithm to determine a fuzzy graph's sigma index is covered in this section.

Algorithm 12.3.1. Let $G = (\sigma, \mu)$ be a fuzzy graph with $\sigma^* = \{w_1, w_2, \cdots, w_n\}$.

1. Construct a matrix $A = [a_{ij}]$ with $a_{ij} = \mu(w_i w_j)$.
2. Let d_i be the sum of entries in the i^{th} row.
3. Make a matrix T such that t_{ij} is the square of difference of d_i and d_j, if $a_{ij} > 0$ and $j > i$ and t_{ij} is equal to zero otherwise.
4. Then $\mathcal{S}(G) = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij}$.

Illustration of Algorithm: Let $G = (\sigma, \mu)$ be a fuzzy graph in Fig. 12.8 with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ and $\mu(w_1w_2) = 0.4$, $\mu(w_1w_3) = 0.3$, $\mu(w_2w_3) = 0.2$, $\mu(w_2w_4) = 0.6$, $\mu(w_2w_6) = 0.4$, $\mu(w_3w_5) = 0.1$, $\mu(w_3w_7) = 0.3$, $\mu(w_4w_6) = 0.3$, $\mu(w_5w_7) = 0.3$, $\mu(w_6w_7) = 0.4$, $\mu(w_6w_8) = 0.1$, $\mu(w_7w_8) = 0.2$.

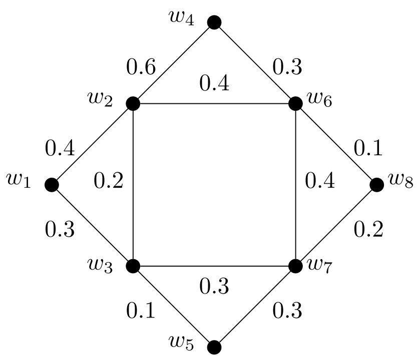

FIGURE 12.8

Illustration for Algorithm 12.3.1.

Step 1: The matrix corresponding to the given fuzzy graph is

$$A = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{array} \\ \begin{pmatrix} 0 & 0.4 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.2 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0.3 & 0.2 & 0 & 0 & 0.1 & 0 & 0.3 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0.4 & 0 & 0.3 & 0 & 0 & 0.4 & 0.1 \\ 0 & 0 & 0.3 & 0 & 0.3 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0 \end{pmatrix} \end{array}$$

Table 12.3.1 Degree of vertices.

vertex	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
d(vertex)	0.7	1.6	0.9	0.9	0.4	1.2	1.2	0.3

Step 2: Calculate $d(i)$ of different vertices. For that, we take the sum of each row in matrix discussed in step 1. The result of the step is given in Table 12.3.1.

Step 3: The matrix T formed by finding square of difference of d_i and d_j, if $a_{ij} > 0$ and $j > i$ and putting zero otherwise.

$$T = \begin{array}{c c} & \begin{array}{cccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{array} \\ \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{array} & \begin{pmatrix} 0 & 0.81 & 0.04 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.49 & 0.49 & 0 & 0.16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 & 0.09 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.09 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.64 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.81 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.81 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step 4: Now the sigma index is the sum of all entries in T. i.e., $\mathcal{S}(G) = 0.81 + 0.04 + 0.49 + 0.49 + 0.16 + 0.25 + 0.09 + 0.09 + 0.64 + 0.81 + 0.81 = 4.68$.

12.4 Application

Financial experts advocate for the creation of a system where money can grow passively. They argue that, apart from investing in gold and real estate, it is advantageous to also invest in the stock market. The stock market provides a platform for trading company shares. While investing in gold offers slow but relatively low-risk growth, selling land for profit typically requires a minimum of three years. In contrast, investing in the stock market allows for flexible withdrawals and the potential for exponential growth. Even with modest returns, the initial value of shares tends to increase over time. Success stories of individuals who have profited from buying shares of companies like Apple and Wipro are well-known. To engage in stock market investments, a Demat account is necessary, similar to a savings account for holding shares instead of money. If you have a Demat account, you may face the dilemma of selecting which companies' shares to purchase for maximum profit. In this situation, standard deviation can be employed to compare the shares of different companies and determine which ones are beneficial to buy.

Let us take an illustration to understand this concept. Imagine observing the share prices of two distinct companies over the course of a week in the Indian stock market. The market operates from 9 a.m. to 5 p.m., Monday through Friday. During these 8

Table 12.4.1 Share prices of company A and company B.

Day	Company A	Company B
Monday	130.15	128.40
Tuesday	132.60	132.90
Wednesday	131.35	127.04
Thursday	130.40	133.60
Friday	130.10	132.60

Table 12.4.2 Fuzzified share prices of company A and company B.

Day	Vertex	Company A	Company B
Monday	w_1	0.9815	0.9610
Tuesday	w_2	1	0.9947
Wednesday	w_3	0.9905	0.9508
Thursday	w_4	0.9834	1
Friday	w_5	0.9811	0.9925

hours, the share prices of the companies will not remain constant; they will fluctuate in response to the prevailing market conditions. This means that the share prices may experience both upward and downward movements. Shares of companies that exhibit significant fluctuations are often categorized as risky shares, as their prices can either rise rapidly or decline dramatically.

The opening price for one week of two companies A and B is given in the below Table 12.4.1. The data is fuzzified in Table 12.4.2 and the fuzzy graph representation is given in Figs. 12.9 and 12.10. Here vertices w_i represent the weekdays. Vertex W is the mean vertex. It is actually a dummy vertex that does not have a physical existence. But the average degree of the vertex is the mean of the data in Table 12.4.2 and edge $w_i W$ represents the contribution of vertex w_i to the mean of the stock.

Vertex W is the mean vertex. Edge $w_i W$ represents the contribution of vertex w_i to the mean of the stock.

Determining the most favorable company can be challenging when we calculate the average price for each company and obtain a result of 130.9. However, by calculating the standard deviation, we can differentiate between Company A and Company B. Company A has a standard deviation of 1.07, while Company B has a higher standard deviation of 2.97. From the definition of the average sigma index, we can find that the average sigma index of a star is similar to its standard deviation. In this case, Company A has an average sigma index of 2.5×10^{-4}, and Company B has an average sigma index of 2×10^{-2}, as determined using Theorem 12.2.9.

A low standard deviation or average sigma index indicates that the values are closer to the central data point, suggesting that they are closer to the mean value. Conversely, the large average sigma index of Company B implies that the values deviate significantly from the mean. Consequently, the share price of Company B is subject

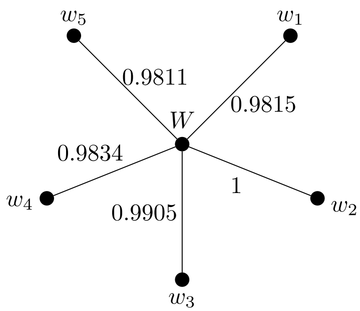

FIGURE 12.9

Fuzzy graph representation of Company A.

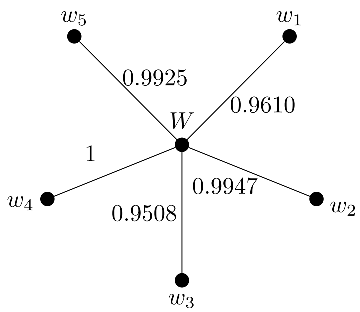

FIGURE 12.10

Fuzzy graph representation of Company B.

to significant fluctuations, making it a high-risk investment. Similar conclusions can be drawn from the average sigma index of the provided data. After analyzing the data, we can conclude that purchasing stocks from Company A are the preferable option.

12.5 Exercises

1. Calculate the sigma index of $G = (\sigma, \mu)$ with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$, $\sigma(x) = 1$ for every $x \in \sigma^*$ and $\mu(w_1w_2) = 0.4$, $\mu(w_2w_3) = 0.6$, $\mu(w_2w_5) = 0.5$, $\mu(w_3w_4) = 0.7\mu(w_4w_5) = 0.5$, $\mu(w_4w_6) = 0.4$, $\mu(w_4w_7) = 0.6$, $\mu(w_5w_6) = 0.6$.

2. Show that sigma index of a proper fuzzy subgraph need not be less than that of its mother graph. Identify a family of fuzzy graphs where this is true.

3. Calculate the average sigma index of the fuzzy graph given in 1.

CHAPTER

Banhatti indices[*]

13

A topological index of a graph is a numerical value connected with its structure. There are several such indices studied in the past related to degree, spectrum, and distance characteristics. The first and second K-Banhatti indices, introduced by Kulli involving vertex and edge degrees, have attained significant interest, due to their applications in different fields. This chapter tries to discuss first and second K-Banhatti indices of fuzzy graphs. Several bounds for the index are presented. Also Banhatti indices of various structures like cycles, complete fuzzy graphs, and complete bipartite graphs are provided. This chapter totally depends on [55].

13.1 First and second K-Banhatti indices

It was Kulli, an Indian mathematician who introduced the concept of Banhatti indices in 2016. A series of related indices were found as a consequence, including K-hyper-Banhatti Indices, Banhatti–Sombor indices, multiplicative-K-Banhatti indices, and multiplicative-K-hyper-Banhatti indices. The fuzzy graph versions of these indices are given in Definitions 13.1.4 and 13.1.5. The vertex membership value $\sigma(w)$ is taken as one for every vertex w, unless otherwise stated.

Definition 13.1.1. The **first K-Banhatti index** of a graph G is defined as $\mathcal{B}_1(G) = \sum_{ew}(deg_G(e) + deg_G(w))$, where ew means that the vertex w and edge e are incident in G, $deg_G(e)$ is the degree of the edge $e = wp$ in graph G and $deg_G(w)$ is the degree of the vertex w in graph G. The **second K-Banhatti index** of a graph G is defined as $\mathcal{B}_1(G) = \sum_{ew}(deg_G(e) \times deg_G(w))$, where ew means that the vertex w and edge e are incident in G, $deg_G(e)$ is the degree of the edge $e = wp$ in graph G and $deg_G(w)$ is the degree of the vertex w in graph G [59].

Definition 13.1.2. The **corona product** of two graphs $G = (W_1, \sigma_1, \mu_1)$ and $H = (W_2, \sigma_2, \mu_2)$ denoted by $G * H = (W, \sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$; is the graph obtained by taking one copy of G of order n and n copies of H, and then joining by an edge the

[*] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

i-th vertex of G to every vertex in the i-th copy of H. Let σ_i be a fuzzy subset of W_i and let μ_i be a fuzzy subset of $E_i, i = 1, 2$. The **corona product** [49] defined by the fuzzy subset $\sigma_1 \circ \sigma_2$ of $W = W_1 \cup |\sigma_1^*|W_2$ and $\mu_1 \circ \mu_2$ of $E = E_1 \cup |\sigma_1^*|E_2 \cup E'$ as follows:

$$(\sigma_1 \circ \sigma_2)(w) = (\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_2)(w) \text{ for all } w \in W.$$

$$(\mu_1 \circ \mu_2)(w) = \begin{cases} (\mu_1 \cup \mu_2 \cup \cdots \cup \mu_2)(wp) & \text{for all } wp \in E - E' \\ \sigma_1(w) \wedge \sigma_2(p) & wp \in E' \end{cases},$$

where E' is the set of all edges joining by an edge the i-th vertex of G to every vertex in the i-th copy of H.

Definition 13.1.3. The **coalescence of fuzzy graphs** [54] $G_1 = (W_1, \sigma_1, \mu_1)$ and $G_2 = (W_2, \sigma_2, \mu_2)$ via w_{G_1} and w_{G_2} is the graph obtained from G_1 and G_2 by identifying $w_{G_1} \in \sigma^*(G_1)$ and $w_{G_2} \in \sigma^*(G_2)$ in a vertex labeled w, denoted by $(G_1.G_2)(w_{G_1}, w_{G_2} : w)(\sigma_1.\sigma_2, \mu_1.\mu_2)$ with

$$\sigma_1.\sigma_2(p) = \begin{cases} \sigma_1(p) & \text{if} \quad p \in W_1, p \neq w \\ \sigma_2(p) & \text{if} \quad p \in W_2, p \neq w \\ \sigma_1(w_{G_1}) \vee \sigma_2(w_{G_2}) & \text{if} \quad p = w \end{cases},$$

$$\mu_1.\mu_2(p_1p_2) = \begin{cases} \mu_1(p_1p_2) & \text{if} \quad p_1p_2 \in E(G_1) \\ \mu_2(p_1p_2) & \text{if} \quad p_1p_2 \in E(G_2) \end{cases},$$

where $\sigma_1 \circ \sigma_2$ is a fuzzy subset of $W = (W_1 \setminus \{w_{G_1}\}) \cup (W_2 \setminus \{w_{G_2}\}) \cup \{w\}$ and $\mu_1 \circ \mu_2$ is a fuzzy subset of $E = E(G_1) \cup E(G_2)$.

Definition 13.1.4. The **first K-Banhatti index of a fuzzy graph** $G = (\sigma, \mu)$ denoted by $\mathcal{B}_1(G)$ is defined as $\mathcal{B}_1(G) = \sum_{ew}(d(e) + d(w))$, where $d(e)$ is the fuzzy degree of the edge $e = wp$, $d(w)$ is the fuzzy degree of the vertex w and ew means that the vertex w and edge e are incident in G.

Definition 13.1.5. The **second K-Banhatti index, of a fuzzy graph** $G = (\sigma, \mu)$ denoted by $\mathcal{B}_2(G)$ is defined as $\mathcal{B}_2(G) = \sum_{ew}(d(e) \times d(w))$, where $d(e)$ is the fuzzy degree of the edge $e = wp$, $d(w)$ is the fuzzy degree of the vertex w and ew means that the vertex w and edge e are incident in G.

In both Definitions 13.1.4 and 13.1.5, the summations are over all vertex-edge pairs. For vertex w, all combinations of all wp's incident at w are considered. An illustration is provided below.

Example 13.1.6. Consider the fuzzy graph $G = (\sigma, \mu)$ given in Fig. 13.1, with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ and $\mu(w_1w_2) = 0.3$, $\mu(w_1w_3) = 0.1$, $\mu(w_2w_3) = 0.4$, $\mu(w_2w_4) = 0.3$, $\mu(w_2w_5) = 0.5$, $\mu(w_3w_6) = 0.6$, $\mu(w_3w_7) = 0.2$, $\mu(w_5w_6) = 0.1$.

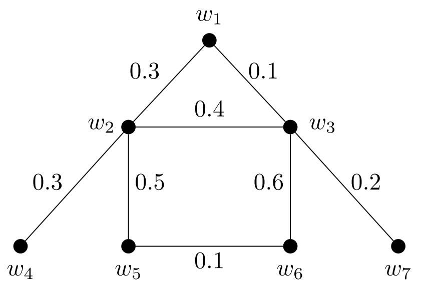

FIGURE 13.1

A fuzzy graph G with $\mathcal{B}_1(G) = 28.1$ and $\mathcal{B}_2(G) = 13.07$.

Table 13.1.1 Degree of vertices and edges of G.

vertex		w_1	w_2	w_3	w_4	w_5	w_6	w_7
d(vertex)		0.4	1.5	0.7	0.3	0.6	0.7	0.2
edge	w_1w_2	w_1w_3	w_2w_3	w_2w_4	w_2w_5	w_3w_6	w_3w_7	w_5w_6
d(edge)	1.3	0.9	1.4	1.2	1.1	0.2	0.5	1.1

Here we have seven vertices and eight edges. First, we have to determine the degree of these vertices and edges. We can see that there are 16 vertex-edge combinations. To find the first K-Banhatti index, we must first add the degree of an edge and a vertex if they are adjacent and later add all those cases together. To find the second K-Banhatti index, we must first take the product of the degree of an edge and a vertex when they are adjacent and then sum all those cases. From Table 13.1.1 and Table 13.1.2, we can see the corresponding calculations. Finally, we obtain the first K-Banhatti index of G as 28.1 and the second K-Banhatti index of G as 13.07.

Next proposition gives the bounds for Banhatti indices for a fuzzy graph with $|\sigma^*| = n$ and $|\mu^*| = m$.

Proposition 13.1.7. *For a fuzzy graph G with $|\sigma^*| = n$, $|\mu^*| = m$ we have*

1. $0 \leq \mathcal{B}_1(G) \leq 2m(3n-5)$.
2. $0 \leq \mathcal{B}_2(G) \leq 4m(n^2-3n+2)$.

Proof. Consider the fuzzy graph $G = (\sigma, \mu)$. If $\mu^* = \phi$, then $d(w_i)$ and $d(w_iw_j)$ are zero for all vertices, w_i and edges, w_iw_j in G^*. Then $\mathcal{B}_1(G) = \sum_{ew}[d_G(e) + d_G(w)] = 0$ and $\mathcal{B}_2(G) = \sum_{ew}[d_G(e) \times d_G(w)] = 0$. Since there does not exist any vertex or edge with negative membership value $\mathcal{B}_1(G)$ and $\mathcal{B}_2(G)$ are always greater than zero. If $|\mu^*| > 0$, then $0 < d(w_i) \leq n-1$ and $0 < d(w_iw_j) \leq n-1+n-1-2 = 2n-4$ for at least some vertices and edges, i.e., $0 < \mathcal{B}_1(G) = \sum_{ew}[d_G(e) + d_G(w)] \leq \sum_{ew} n-1+2n-4 = \sum_{ew} 3n-5 = 2m(3n-5)$ and $0 <$

Table 13.1.2 Calculation of first and second K-Banhatti indices.

edge(ab)vertex(a)	$d(ab)+d(a)$	$d(ab)\times d(a)$
$(w_1w_2)w_1$	1.7	0.52
$(w_1w_2)w_2$	2.8	1.95
$(w_1w_3)w_1$	1.3	0.36
$(w_1w_3)w_3$	1.6	0.63
$(w_2w_3)w_2$	2.9	2.1
$(w_2w_3)w_3$	2.1	0.98
$(w_2w_4)w_2$	2.7	1.8
$(w_2w_4)w_4$	1.5	0.36
$(w_2w_5)w_2$	2.6	1.65
$(w_2w_5)w_5$	1.7	0.66
$(w_3w_6)w_3$	0.9	0.14
$(w_3w_6)w_6$	0.9	0.14
$(w_3w_7)w_3$	1.2	0.35
$(w_3w_7)w_7$	0.7	0.1
$(w_5w_6)w_5$	1.7	0.66
$(w_5w_6)w_6$	1.8	0.77
	$\mathcal{B}_1(G)=28.1$	$\mathcal{B}_2(G)=13.07$

$\mathcal{B}_2(G)=\sum_{ew}[d_G(e)\times d_G(w)]\leq\sum_{ew}(n-1)\times(2n-4)=4m(n^2-3n+2)$. Now consider the complete fuzzy graph with all vertices having membership value one, then $\mathcal{B}_1(G)=\sum_{ew}[d_G(e)+d_G(w)]=\sum_{ew}n-1+2n-4=\sum_{ew}3n-5=2m(3n-5)$ and $\mathcal{B}_2(G)\leq\sum_{ew}[d_G(e)\times d_G(w)]=\sum_{ew}(n-1)\times(2n-4)=4m(n^2-3n+2)$. Therefore $0\leq\mathcal{B}_1(G)\leq 2m(3n-5)$ and $0\leq\mathcal{B}_2(G)\leq 4m(n^2-3n+2)$. ■

The following proposition and corollary give relationship between the first and second K-Banhatti indices a given fuzzy graph and its partial fuzzy subgraphs.

Proposition 13.1.8. *If $H=(\tau,\nu)$ is a partial fuzzy subgraph of $G=(\sigma,\mu)$, then $\mathcal{B}_1(H)\leq\mathcal{B}_1(G)$ and $\mathcal{B}_2(H)\leq\mathcal{B}_2(G)$.*

Proof. Suppose $H=(\tau,\nu)$ be a partial fuzzy subgraph of $G=(\sigma,\mu)$, with $\sigma^*=\{w_1,w_2,\cdots,w_n\}$. Let w_i be an arbitrary vertex in τ^*, then $\nu(w_iw_j)\leq\mu(w_iw_j)$ for all other vertices w_j in τ^*. Which gives $d_H(w_i)\leq d_G(w_i)$ for all vertices w_i in τ^*. Now consider the degree of an arbitrary edge w_iw_j, $d_H(w_iw_j)=d_H(w_i)+d_H(w_j)-2\times\mu_H(w_iw_j)\leq d_G(w_i)+d_G(w_j)-2\times\mu_G(w_iw_j)=d_G(w_iw_j)$. Hence, $d_H(w_i)+d_H(w_iw_j)\leq d_G(w_i)+d_G(w_iw_j)$ and $d_H(w_i)\times d_H(w_iw_j)\leq d_G(w_i)\times d_G(w_iw_j)$. Therefore $\mathcal{B}_1(H)=\sum_{ew}[d_H(e)+d_H(w)]=\sum_{i=1}^{n}\sum_{j\in N(w_i)}[d_H(w_iw_j)+$

$d_H(w_i)] \leq \sum_{i=1}^{n} \sum_{j \in N(w_i)} [d_G(w_i w_j) + d_G(w_i)] = \mathcal{B}_1(G)$, i.e., $\mathcal{B}_1(H) \leq \mathcal{B}_1(G)$. Also,

$$\mathcal{B}_2(H) = \sum_{ew} [d_H(e) \times d_H(w)] = \sum_{i=1}^{n} \sum_{j \in N(w_i)} [d_H(w_i w_j) \times d_H(w_i)]$$

$$\leq \sum_{i=1}^{n} \sum_{j \in N(w_i)} [d_G(w_i w_j) \times d_G(w_i)] = \mathcal{B}_2(G), \text{ i.e., } \mathcal{B}_2(H) \leq \mathcal{B}_2(G). \qquad \blacksquare$$

Corollary 13.1.9. *For a fuzzy graph $G = (\sigma, \mu)$ with vertex set σ^* and complete fuzzy graph $G' = (\sigma', \mu')$ spanned by σ^*, we have $0 \leq \mathcal{B}_1(G) \leq \mathcal{B}_1(G')$ and $0 \leq \mathcal{B}_2(G) \leq \mathcal{B}_2(G')$.*

A number of fuzzy graph structures are considered in the following theorems. Cycles, saturated fuzzy cycles, paths, complete fuzzy graphs, complete bipartite graphs, and stars are considered. The variables t, t_i, or t_{ij} are considered within the range of zero to one.

Theorem 13.1.10. *For a cycle C_n with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_i w_{i+1}$ having $\mu(e_i) = t_i > 0$, we have*

1. $\mathcal{B}_1(C_n) = 8 \sum_{i=1}^{n} t_i$.
2. $\mathcal{B}_2(C_n) = 2 \left(\sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} t_i t_{i+2} \right) + 4 \sum_{i=1}^{n} t_i t_{i+1}$ *where i is taken under modulo n.*

Proof. Let C_n be a cycle as stated in the theorem. Consider an arbitrary vertex w_i. Fuzzy degree of vertex $w_i, d(w_i) = t_{i-1} + t_i$. Fuzzy degree of edge $w_{i-1}w_i$, $d(w_{i-1}w_i) = t_{i-2} + t_i$ and fuzzy degree of edge $w_i w_{i+1}$, $d(w_i w_{i+1}) = t_{i-1} + t_{i+1}$.

1. The first K-Banhatti index is given as, $\mathcal{B}_1(C_n) = \sum_{ew} [d_{C_n}(e) + d_{C_n}(w)] = \sum_{i=1}^{n} \sum_{j=1}^{2} [d_{C_n}(e_j) + d_{C_n}(w_i)] = \sum_{i=1}^{n} [d_{C_n}(e_1) + d_{C_n}(w_i)] + \sum_{i=1}^{n} [d_{C_n}(e_2) + d_{C_n}(w_i)] = \sum_{i=1}^{n} [(t_{i-2} + t_i) + (t_{i-1} + t_i)] + \sum_{i=1}^{n} [(t_{i-1} + t_{i+1}) + (t_{i-1} + t_i)] = 8 \sum_{i=1}^{n} t_i$.
2. The second K-Banhatti index is given as,

$$\mathcal{B}_2(C_n) = \sum_{ew} [d_{C_n}(e) \times d_{C_n}(w)]$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{2} [d_{C_n}(e_j) \times d_{C_n}(w_i)]$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}[d_{C_n}(e_1) \times d_{C_n}(w_i)] + \sum_{i=1}^{n}[d_{C_n}(e_2) \times d_{C_n}(w_i)] \\
&= \sum_{i=1}^{n}[(t_{i-2}+t_i) \times (t_{i-1}+t_i)] + \sum_{i=1}^{n}[(t_{i-1}+t_{i+1}) \times (t_{i-1}+t_i)] \\
&= \sum_{i=1}^{n}[t_{i-2}t_{i-1} + t_{i-2}t_i + t_i t_{i-1} + t_i^2] \\
&\quad + \sum_{i=1}^{n}[t_{i-1}^2 + t_{i-1}t_i + t_{i+1}t_{i-1} + t_{i+1}t_i] \\
&= 2\left(\sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} t_i t_{i+2}\right) + 4\sum_{i=1}^{n} t_i t_{i+1}. \qquad \blacksquare
\end{aligned}
$$

Corollary 13.1.11. *Let C_n be a cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_j = w_j w_{j+1}$ having $\mu(e_j) = t_j > 0$. Let C_{in}, $i \in \mathbb{N}$ be cycles with vertex set $\{w_1, w_2, \cdots, w_{in}\}$ and edge set $\{e_1, e_2, \cdots, e_{in}\}$ where $e_j = w_j w_{j+1}$ having $\mu(e_j) = t_j$, for $1 \le j \le n$, the same t_j mentioned in C_n and $\mu(e_k) = \mu(e_l)$, $k = l \mod n$. Then $B_1(C_{in}) = i \times \mathcal{B}_1(C_n)$ and $B_2(C_{in}) = i \times \mathcal{B}_2(C_n)$.*

Corollary 13.1.12. *Let C_n be a saturated fuzzy cycle with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_j = w_j w_{j+1}$. Suppose that all its β-edges have weight x and α-edges have weight $t_1, t_2, \cdots, t_{\frac{n}{2}}$, i.e., $\mu(e_{2i+1}) = x$, for $0 \le i \le \frac{n-2}{2}$ and $\mu(e_{2i}) = t_i$, for $1 \le i \le \frac{n}{2}$, then*

$$
\mathcal{B}_1(C_n) = 8\left[\sum_{i=1}^{\frac{n}{2}} t_i + \frac{n}{2}x\right],
$$

$$
\mathcal{B}_2(C_n) = 2\left(\sum_{i=1}^{\frac{n}{2}} t_i^2 + \sum_{i=1}^{\frac{n}{2}} t_i t_{i+1} + nx^2\right) + 8\sum_{i=1}^{\frac{n}{2}} t_i x
$$

where i is taken under modulo $\frac{n}{2}$.

Proof. Let C_n be a cycle as stated in the theorem. Fuzzy degree of $w_i's$ are given as, $d(w_{2i}) = d(w_{2i+1}) = t_i + x$ for $1 \le i \le \frac{n}{2}$. The fuzzy degree of the edge $w_{2i}w_{2i+1}$, $d(w_{2i}w_{2i+1}) = 2x$ and fuzzy degree of the edge $w_{2i+1}w_{2(i+1)}$, $d(w_{2i+1}w_{2(i+1)}) = t_i + t_{i+1}$.

The first K-Banhatti index is given as,

$$
\begin{aligned}
\mathcal{B}_1(C_n) &= \sum_{ew}[d_{C_n}(e) + d_{C_n}(w)] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{2}[d_{C_n}(e_j) + d_{C_n}(w_i)] \\
&= \sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{2}[d_{C_n}(e_j) + d_{C_n}(w_{2i})] + \sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{2}[d_{C_n}(e_j) + d_{C_n}(w_{2i+1})] \\
&= \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_1) + d_{C_n}(w_{2i})] + \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_2) + d_{C_n}(w_{2i})] \\
&\quad + \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_1) + d_{C_n}(w_{2i+1})] + \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_2) + d_{C_n}(w_{2i+1})] \\
&= \sum_{i=1}^{\frac{n}{2}}[(t_{i-1} + t_i) + (t_i + x)] + \sum_{i=1}^{\frac{n}{2}}[2x + (t_i + x)] + \sum_{i=1}^{\frac{n}{2}}[2x + (t_i + x)] \\
&\quad + \sum_{i=1}^{\frac{n}{2}}[(t_i + t_{i+1}) + (t_i + x)] \\
&= 8\left[\sum_{i=1}^{\frac{n}{2}} t_i + \frac{n}{2}x\right].
\end{aligned}
$$

The second K-Banhatti index is given as,

$$
\begin{aligned}
\mathcal{B}_2(C_n) &= \sum_{ew}[d_{C_n}(e) \times d_{C_n}(w)] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{2}[d_{C_n}(e_j) \times d_{C_n}(w_i)] \\
&= \sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{2}[d_{C_n}(e_j) \times d_{C_n}(w_{2i})] + \sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{2}[d_{C_n}(e_j) \times d_{C_n}(w_{2i+1})] \\
&= \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_1) \times d_{C_n}(w_{2i})] + \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_2) \times d_{C_n}(w_{2i})] \\
&\quad + \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_1) \times d_{C_n}(w_{2i+1})] + \sum_{i=1}^{\frac{n}{2}}[d_{C_n}(e_2) \times d_{C_n}(w_{2i+1})]
\end{aligned}
$$

$$= \sum_{i=1}^{\frac{n}{2}}[(t_{i-1}+t_i)\times(t_i+x)] + \sum_{i=1}^{\frac{n}{2}}[2x\times(t_i+x)] + \sum_{i=1}^{\frac{n}{2}}[2x\times(t_i+x)]$$
$$+ \sum_{i=1}^{\frac{n}{2}}[(t_i+t_{i+1})\times(t_i+x)]$$
$$= \sum_{i=1}^{\frac{n}{2}}[t_{i-1}t_i + t_{i-1}x + t_i^2 + t_ix] + \sum_{i=1}^{\frac{n}{2}}[2xt_i + 2x^2] + \sum_{i=1}^{\frac{n}{2}}[2xt_i + 2x^2]$$
$$+ \sum_{i=1}^{\frac{n}{2}}[t_i^2 + t_ix + t_{i+1}t_i + t_{i+1}x]$$
$$= 2\left(\sum_{i=1}^{\frac{n}{2}} t_i^2 + \sum_{i=1}^{\frac{n}{2}} t_it_{i+1} + nx^2\right) + 8\sum_{i=1}^{\frac{n}{2}} t_ix. \qquad \blacksquare$$

Suppose that every α strong edges have strength s and every β strong edges have strength t in a saturated fuzzy cycle C_n, then $\mathcal{B}_1(C_n) = 4n(s+t)$ and $\mathcal{B}_2(C_n) = 2n(s^2+t^2+2st)$.

Corollary 13.1.13. *Let P be a path with $\sigma^* = \{w_1, w_2, \cdots, w_n\}$ and $\mu(w_iw_{i+1}) = t_i$, $1 \le i \le n-1$, $n > 2$. Then,*

1. $\mathcal{B}_1(P_n) = 5t_1 + 8\sum_{i=2}^{n-2} t_i + 5t_{n-1}$.

2. $\mathcal{B}_2(P_n) = t_1^2 + t_{n-1}^2 + 2\left(\sum_{i=2}^{n-2} t_i^2 + \sum_{i=1}^{n-3} t_it_{i+2}\right) + 4\sum_{i=1}^{n-2} t_it_{i+1}$.

Theorem 13.1.14. *Let $G = (\sigma, \mu)$ be a CFG with $\sigma^* = \{w_1, w_2, \cdots, w_n\}$ such that $t_1 \le t_2 \le \cdots \le t_n$, where $t_i = \sigma(w_i)$, $1 \le i \le n$. Then $\mathcal{B}_1(G) = (6n - 10)\,[(n-1)t_1 + (n-2)t_2 + \cdots + t_{n-1}]$.*

Proof. Let G be a complete fuzzy graph as stated in the theorem. Consider an arbitrary vertex w_i, it has $n-i$ edges of weight t_i incident at w_i and remaining $(n-1)-(n-i) = i-1$ edges of weight t_1 to t_{i-1}, by construction, i.e., $d(w_i) = (n-i)t_i + \sum_{k=1}^{i-1} t_k$. Now consider an arbitrary edge w_iw_j, its degree is $(n-i)t_i + \sum_{k=1}^{i-1} t_k + (n-j)t_j + \sum_{k=1}^{j-1} t_k - 2\mu(w_iw_j)$. Therefore the first K-Banhatti index of a complete fuzzy graph is given by,

$$
\begin{aligned}
\mathcal{B}_1(G) &= \sum_{ew}[d_G(e) + d_G(w)] \\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}[d_G(w_iw_j) + d_G(w_i)] \\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\Bigg[(n-i)t_i + \sum_{k=1}^{i-1}t_k + (n-j)t_j \\
&\qquad + \sum_{k=1}^{j-1}t_k - 2\mu(w_iw_j) + (n-i)t_i + \sum_{k=1}^{i-1}t_k\Bigg] \\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[(n-i)t_i + \sum_{k=1}^{i-1}t_k + (n-j)t_j + \sum_{k=1}^{j-1}t_k + (n-i)t_i + \sum_{k=1}^{i-1}t_k\right] \\
&\qquad - 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j) \\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[2(n-i)t_i + 2\sum_{k=1}^{i-1}t_k + (n-j)t_j + \sum_{k=1}^{j-1}t_k\right] - 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left[2(n-i)t_i + 2\sum_{k=1}^{i-1}t_k + (n-j)t_j + \sum_{k=1}^{j-1}t_k\right] \\
&\qquad - 3\left[\sum_{i=1}^{n}\left((n-i)t_i - \sum_{k=1}^{i-1}t_k\right)\right] - 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j) \\
&= \sum_{i=1}^{n}2n\left[(n-i)t_i + \sum_{k=1}^{i-1}t_k\right] + \sum_{j=1}^{n}n\left[(n-j)t_j + \sum_{k=1}^{j-1}t_k\right] \\
&\qquad - 3\left[\sum_{i=1}^{n}\left((n-i)t_i - \sum_{k=1}^{i-1}t_k\right)\right] - 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j) \\
&= \sum_{i=1}^{n}3n\left[(n-i)t_i + \sum_{k=1}^{i-1}t_k\right] - 3\left[\sum_{i=1}^{n}\left((n-i)t_i - \sum_{k=1}^{i-1}t_k\right)\right]
\end{aligned}
$$

$$-2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j)$$

$$=(3n-3)\sum_{i=1}^{n}\left[(n-i)t_i+\sum_{k=1}^{i-1}t_k\right]-2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j) \qquad \text{(i)}$$

Now we can find $\sum_{i=1}^{n}\left[(n-i)t_i+\sum_{k=1}^{i-1}t_k\right]$ and $\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j)$ separately.

$$\begin{aligned}\sum_{i=1}^{n}\left[(n-i)t_i+\sum_{k=1}^{i-1}t_k\right] &= (n-1)t_1+(n-2)t_2+\cdots+(n-(n-1))t_{n-1}\\ &\quad+(t_1)+(t_1+t_2)+\cdots+(t_1+t_2+\cdots+t_{n-1})\\ &=2[(n-1)t_1+(n-2)t_2+\cdots+t_{n-1}] \qquad \text{(ii)}\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu(w_iw_j) &= 2 \text{ times sum of all edge membership values}\\ &=2[(n-1)t_1+(n-2)t_2+\cdots+t_{n-1}] \qquad \text{(iii)}\end{aligned}$$

Substituting equation (ii) and Eq. (iii) in Eq. (i) gives

$$\begin{aligned}\mathcal{B}_1(G) &= (3n-3)\times 2[(n-1)t_1+(n-2)t_2+\cdots+t_{n-1}]\\ &\quad-[2\times 2[(n-1)t_1+(n-2)t_2+\cdots+t_{n-1}]]\\ &=(6n-10)\,[(n-1)t_1+(n-2)t_2+\cdots+(n-(n-1))t_{n-1}].\end{aligned}$$ ■

Theorem 13.1.15. *Let $G=(\sigma,\mu)$ be a complete bipartite graph with $\sigma^*=\{w_1,w_2,\cdots,w_m,p_1,p_2,\cdots p_n\}$ such that $\mu(w_ip_j)=t_{ij}$, for $1\leq i\leq m$, $1\leq j\leq n$. Then*

1. $\mathcal{B}_1(G)=[3(m+n)-4]\left[\sum_{i=1}^{m}\sum_{j=1}^{n}t_{ij}\right]$.

2. $\mathcal{B}_2(G)=(n-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}t_{ij}t_{ik}+(m-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m}t_{ij}t_{rj}+2\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m}\sum_{k=1}^{n}t_{ij}t_{rk}$.

Proof. Let G be a complete bipartite graph as stated in the theorem. The vertices of G can be divided into two independent sets, $W=\{w_1,w_2,\cdots,w_m\}$ and

$P = \{p_1, p_2, \cdots p_n\}$. Take an arbitrary vertex $w_i \in W$, its degree is $\sum_{k=1}^{n} t_{ik}$. Next consider an arbitrary vertex $p_j \in P$, its degree is $\sum_{r=1}^{m} t_{rj}$. Now consider an arbitrary edge $w_i p_j$ connecting W and P, its degree is $\sum_{k=1}^{n} t_{ik} + \sum_{r=1}^{m} t_{rj} - 2t_{ij}$. Therefore the first K-Banhatti index is given as

$$
\begin{aligned}
\mathcal{B}_1(G) &= \sum_{ew}[d_G(e) + d_G(w)] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\big[d(w_i) + d(w_i p_j)\big] + \sum_{j=1}^{n}\sum_{i=1}^{m}\big[d(p_j) + d(w_i p_j)\big] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[2\sum_{k=1}^{n} t_{ik} + \sum_{r=1}^{m} t_{rj} - 2t_{ij}\right] + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[2\sum_{r=1}^{m} t_{rj} + \sum_{k=1}^{n} t_{ik} - 2t_{ij}\right] \\
&= \sum_{i=1}^{m} 2n \sum_{k=1}^{n} t_{ik} + \sum_{j=1}^{n} m \sum_{r=1}^{m} t_{rj} - 2\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} + \sum_{j=1}^{n} 2m \sum_{r=1}^{m} t_{rj} \\
&\quad + \sum_{i=1}^{m} n \sum_{k=1}^{n} t_{ik} - 2\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} \\
&= (2n + m - 2)\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} + (2m + n - 2)\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} \\
&= [3(m+n) - 4]\left[\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}\right].
\end{aligned}
$$

The second K-Banhatti index is given as

$$
\begin{aligned}
\mathcal{B}_2(G) &= \sum_{ew}[d_G(e) \times d_G(w)] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\big[d(w_i) \times d(w_i p_j)\big] + \sum_{j=1}^{n}\sum_{i=1}^{m}\big[d(p_j) \times d(w_i p_j)\big] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} t_{ik}\right)\left(\sum_{k=1}^{n} t_{ik} + \sum_{r=1}^{m} t_{rj} - 2t_{ij}\right)\right] \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\left(\sum_{r=1}^{m} t_{rj}\right)\left(\sum_{r=1}^{m} t_{rj} + \sum_{k=1}^{n} t_{ik} - 2t_{ij}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(t_{i1}+t_{i2}+\cdots+t_{in})\left(t_{i1}+t_{i2}+\cdots+t_{in}+t_{1j}+t_{2j}+\cdots+t_{mj}-2t_{ij}\right)\right] \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\left(\sum_{r=1}^{m} t_{rj}\right)\left(\sum_{r=1}^{m} t_{rj}+\sum_{k=1}^{n} t_{ik}-2t_{ij}\right)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(t_{i1}t_{i1}+t_{i1}t_{i2}+\cdots+t_{i1}t_{in}+t_{i1}t_{1j}+t_{i1}t_{2j}+\cdots+t_{i1}t_{mj}-2t_{i1}t_{ij}\right) \\
&\quad + \left(t_{i2}t_{i1}+t_{i2}t_{i2}+\cdots+t_{i2}t_{in}+t_{i2}t_{1j}+t_{i2}t_{2j}+\cdots+t_{i2}t_{mj}-2t_{i2}t_{ij}\right)+\cdots \\
&\quad + \left(t_{in}t_{i1}+t_{in}t_{i2}+\cdots+t_{in}t_{in}+t_{in}t_{1j}+t_{in}t_{2j}+\cdots+t_{in}t_{mj}-2t_{in}t_{ij}\right) \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\left(\sum_{r=1}^{m} t_{rj}\right)\left(\sum_{r=1}^{m} t_{rj}+\sum_{k=1}^{n} t_{ik}-2t_{ij}\right)\right] \\
&= n\sum_{i=1}^{m}((t_{i1}t_{i1}+t_{i1}t_{i2}+\cdots+t_{i1}t_{in})+(t_{i2}t_{i1}+t_{i2}t_{i2}+\cdots+t_{i2}t_{in})) \\
&\quad + (\cdots+(t_{in}t_{i1}+t_{in}t_{i2}+\cdots+t_{in}t_{in})) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}((t_{i1}t_{1j}+t_{i1}t_{2j}+\cdots+t_{i1}t_{mj})+(t_{i2}t_{1j}+t_{i2}t_{2j}+\cdots+t_{i2}t_{mj}) \\
&\quad + \cdots+\left(+t_{in}t_{1j}+t_{in}t_{2j}+\cdots+t_{in}t_{mj})\right)-2\sum_{i=1}^{m}\sum_{j=1}^{n}(t_{i1}t_{ij}+t_{i2}t_{ij}+\cdots+t_{in}t_{ij}) \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\left(\sum_{r=1}^{m} t_{rj}\right)\left(\sum_{r=1}^{m} t_{rj}+\sum_{k=1}^{n} t_{ik}-2t_{ij}\right)\right] \\
&= (n-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} t_{ij}t_{ik}+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m}\sum_{k=1}^{n} t_{ij}t_{rk} \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\left(\sum_{r=1}^{m} t_{rj}\right)\left(\sum_{r=1}^{m} t_{rj}+\sum_{k=1}^{n} t_{ik}-2t_{ij}\right)\right].
\end{aligned}
$$

After similar calculations for
$\sum_{j=1}^{n}\sum_{i=1}^{m}\left[\left(\sum_{r=1}^{m} t_{rj}\right)\left(\sum_{r=1}^{m} t_{rj}+\sum_{k=1}^{n} t_{ik}-2t_{ij}\right)\right]$ we get

$$
\begin{aligned}
\mathcal{B}_2(G) &= (n-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} t_{ij}t_{ik}+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m}\sum_{k=1}^{n} t_{ij}t_{rk}+(m-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m} t_{ij}t_{rj} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m}\sum_{k=1}^{n} t_{ij}t_{rk}
\end{aligned}
$$

$$= (n-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} t_{ij}t_{ik} + (m-2)\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m} t_{ij}t_{rj} + 2\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{m}\sum_{k=1}^{n} t_{ij}t_{rk}.$$

■

The following corollary finds the first and second K-Banhatti indices of a star graph.

Corollary 13.1.16. *Let $S = (\sigma, \mu)$ be a star, with vertex set $\{w_1, w_2, \cdots, w_m, p_1\}$, where p_1 is the central vertex and edge set $\{e_1, e_2, \cdots, e_m\}$ where $e_i = w_i p_1$ with $\mu(w_i p_1) = t_{i1}$. Then*

1. $\mathcal{B}_1(G) = [3m-1]\left[\sum_{i=1}^{m} t_{i1}\right].$
2. $\mathcal{B}_2(G) = m\sum_{i=1}^{m}\sum_{r=1}^{m} t_{i1}t_{r1} - \sum_{i=1}^{m} t_{i1}t_{i1}.$

Theorem 13.1.17. *For a fuzzy tree G, which is not a tree, with F as its maximum spanning tree, $\mathcal{B}_1(F) < \mathcal{B}_1(G)$ and $\mathcal{B}_2(F) < \mathcal{B}_2(G)$.*

Proof. Let G be a fuzzy graph which is not a tree and F be its maximum spanning tree. There exist at least one vertex w_i with $d_F(w_i) < d_G(w_i)$, since G is not a tree. Similarly, there exist at least an edge $w_k w_j$ such that $d_F(w_k w_j) < d_G(w_k w_j)$. Therefore $\mathcal{B}_1(F) = \sum_{ew}[d_F(e) + d_F(w)] < \sum_{ew}[d_G(e) + d_G(w)] = \mathcal{B}_1(G)$ and $\mathcal{B}_2(F) = \sum_{ew}[d_F(e) \times d_F(w)] < \sum_{ew}[d_G(e) \times d_G(w)] = \mathcal{B}_2(G)$. ■

Remark. Degree of vertices and edges are preserved under isomorphisms. Thus first and second Banhatti indices of isomorphic graphs are the same. The situation is similar in fuzzy graph theory also.

Let σ and τ be fuzzy subsets of a set W. We write $\sigma \supseteq \tau$ if for all $w \in W, \sigma(w) \geq \tau(w)$. If $\sigma \supseteq \tau$ and there exists $w \in W$ such that $\sigma(w) > \tau(w)$, we write $\sigma \supset \tau$.

Definition 13.1.18. Let $H = (W, \tau, \nu)$ be a fuzzy graph. Then a fuzzy graph $G = (W, \sigma, \mu)$ is called a **partial fuzzy supergraph** of G if $\sigma \supseteq \tau$ and $\mu \supseteq \nu$. Similarly, the fuzzy graph $G = (P, \tau, \nu)$ is called a **fuzzy supergraph** of G if $P \supseteq W$, $\tau(w) = \sigma(w)$ for all $w \in W$ and $\nu(wp) = \mu(wp)$ for all $w, p \in W$.

Theorem 13.1.19. *There does not exist any connected fuzzy supergraph with same first or second K-Banhatti index as that of the given graph.*

Proof. Let G be a fuzzy graph. Let H be a fuzzy supergraph of G with an additional vertex p. Then $0 = d_G(p) < d_H(p)$, since p does not belong to G. Consider an arbitrary vertex w in G, w may or may not have an edge with p. Thus

$d_G(w) \le d_H(w)$. Let px be an arbitrary edge in the fuzzy graph H connecting p and G then $0 = d_G(px) < d_H(px) = d_H(p) + d_H(x) - 2\mu_H(px)$. Let wy be an arbitrary edge within G. Then $d_G(wy) = d_G(w) + d_G(y) - 2\mu_G(wy) \le d_H(w) + d_H(y) - 2\mu_H(wy) = d_H(wy)$, even though wy lies within G, w, or y may have an edge with p, which may cause an increase in fuzzy degree for those vertices in H. Therefore $\mathcal{B}_1(G) = \sum_{ew}[d_G(e)+d_G(w)] < \sum_{ew}[d_H(e)+d_H(w)] = \mathcal{B}_1(H)$. Similarly, $\mathcal{B}_2(G) = \sum_{ew}[d_G(e) \times d_G(w)] < \sum_{ew}[d_H(e) \times d_H(w)] = \mathcal{B}_2(H)$.

Now consider the case where H is a fuzzy supergraph formed with an additional edge wp where $w, p \in G$. Consider an arbitrary vertex $x \in G$, if $x \ne w, p$ then $d_G(x) = d_H(x)$. If $x = w, p$ then $d_G(x) \le d_H(x)$. Consider the edge wp, then $0 = d_G(wp) < d_H(wp)$, since wp does not belong to G. Next consider an arbitrary edge $xy \in G, x, y \ne w, p$ then $d_G(xy) = d_G(x) + d_G(y) - 2\mu_G(xy) = d_H(x) + d_H(y) - 2\mu_H(xy) = d_H(xy)$. Now consider an arbitrary edge $xy \in G, x$ or $y = w$ or p then $d_G(xy) = d_G(x) + d_G(y) - 2\mu_G(xy) \le d_H(x) + d_H(y) - 2\mu_H(xy) = d_H(xy)$. Therefore $\mathcal{B}_1(G) = \sum_{ew}[d_G(e)+d_G(w)] < \sum_{ew}[d_H(e)+d_H(w)] = \mathcal{B}_1(H)$. Similarly, $\mathcal{B}_2(G) = \sum_{ew}[d_G(e) \times d_G(w)] < \sum_{ew}[d_H(e) \times d_H(w)] = \mathcal{B}_2(H)$. ■

Proposition 13.1.20 provides the connection between Banhatti indices of a fuzzy graph and its complement. We follow the definition of complement from [108].

Proposition 13.1.20. *Consider a fuzzy cycle $G = (\sigma, \mu)$ having more than 3 vertices with vertex set $\{w_1, w_2, \cdots, w_n\}$ and edge set $\{e_1, e_2, \cdots, e_n\}$ where $e_i = w_i w_{i+1}$ having $\mu(e_i) = t_i > 0$, and $\sigma(w_i) = t$ for all $w_i \in \sigma^*$. Then*

1. $3nt(n-1)(n-3) \le \mathcal{B}_1(G) + \mathcal{B}_1(G^c) \le nt(3n^2 - 8n + 13)$,
2. $2nt^2(n-1)(n-3)^2 \le \mathcal{B}_2(G) + \mathcal{B}_2(G^c) \le 2nt^2(n^3 - 4n^2 + 5n + 2)$

where $G^c = (\sigma^c, \mu^c)$ is the fuzzy complement of the fuzzy graph $G = (\sigma, \mu)$.

Proof. Let G be a fuzzy cycle with its vertices having weight t. Let w_i be an arbitrary vertex, then $d_G(w_i) = t_{i-1} + t_i$, which implies $0 \le d_G(w_i) \le 2t$. Let $w_i w_{i+1}$ be an arbitrary edge from w_i, its degree is, $d_G(w_i w_{i+1}) = t_{i-1} + t_{i+1}$, which implies $0 \le d_G(w_i w_{i+1}) \le 2t$. Therefore $0 \le d_G(w_i) + d_G(w_i w_{i+1}) \le 4t$, implies $0 \le d_G(w_i) + d_G(w_i w_{i+1}) + d_G(w_{i+1}) + d_G(w_i w_{i+1}) \le 8t$, implies $0 \le \mathcal{B}_1(G) \le 8nt$, since there are n such edges. Similarly, $0 \le d_G(w_i) \times d_G(w_i w_{i+1}) \le 4t^2$, implies $0 \le d_G(w_i) \times d_G(w_i w_{i+1}) + d_G(w_{i+1}) \times d_G(w_i w_{i+1}) \le 8t^2$, implies $0 \le \mathcal{B}_2(G) \le 8nt^2$, since there are n such edges. Now consider the fuzzy complement of G. Consider an arbitrary vertex w_i in G^c. Degree of the vertex w_i, $d_{G^c}(w_i) = (n-1)t - (t_{i-1} + t_i)$ which implies $(n-3)t \le d_{G^c}(w_i) \le (n-1)t$. Let $w_i w_{i+1}$ be an arbitrary edge from w_i, its degree is, $d_{G^c}(w_i w_{i+1}) = 2t(n-2) - (t_{i-1} + t_{i+1})$, which implies $2t(n-3) \le d_{G^c}(w_i w_{i+1}) \le 2t(n-2)$. Therefore $3nt - 9t \le d_{G^c}(w_i) + d_{G^c}(w_i w_{i+1}) \le 3nt - 5t$, implies $6nt - 18t \le d_{G^c}(w_i) + d_{G^c}(w_i w_{i+1}) + d_{G^c}(w_{i+1}) + d_{G^c}(w_i w_{i+1}) \le$

$6nt - 10t$, implies $3nt(n-1)(n-3) \leq \mathcal{B}_1(G^c) \leq nt(n-1)(3n-5)$, since there are ${}^nC_2$ such edges. Similarly, $(nt-3t)(2nt-6t) \leq d_{G^c}(w_i) \times d_{G^c}(w_i w_{i+1}) \leq (nt-t)(2nt-4t)$, implies $4t^2(n-3)^2 \leq d_{G^c}(w_i) \times d_{G^c}(w_i w_{i+1}) + d_{G^c}(w_{i+1}) \times d_{G^c}(w_i w_{i+1}) \leq 4t^2(n-1)(n-2)$, implies $2nt^2(n-1)(n-3)^2 \leq \mathcal{B}_2(G) \leq 2nt^2(n-1)^2(n-2)$, since there are ${}^nC_2$ such edges. From the above obtained inequalities it can be found that $3nt(n-1)(n-3) \leq \mathcal{B}_1(G) + \mathcal{B}_1(G^c) \leq nt(3n^2 - 8n + 13)$ and $2nt^2(n-1)(n-3)^2 \leq \mathcal{B}_2(G) + \mathcal{B}_2(G^c) \leq 2nt^2(n^3 - 4n^2 + 5n + 2)$. ■

13.2 Fuzzy graph operations and the K-Banhatti indices

In this section, we look at the first and second K-Banhatti indices of graphs obtained from various fuzzy graph operations. The symbol $G_1.G_2$ represents the coalescence of two fuzzy graphs, G_1 and G_2, while $G_1 \circ G_2$ represents the corona product of two fuzzy graphs G_1 and G_2.

Theorem 13.2.1. *Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$ and $|\sigma_2^*| = n_2$. Let $\{s_1, s_2, \cdots, s_{n_1}\}$ be the vertex set of G_1 and $\{r_1, r_2, \cdots, r_{n_2}\}$ be the vertex set of G_2. Let s_i and r_j be arbitrary vertices from G_1 and G_2. Then the first K-Banhatti index of fuzzy graph coalescence of G_1 and G_2 at s_i and r_j, $\mathcal{B}_1(G_1.G_2)(s_i, r_j : w_*) = \mathcal{B}_1(G_1) + \mathcal{B}_1(G_2) + 3(deg_{G_1}(w_*)d_{G_2}(w_*) + deg_{G_2}(w_*)d_{G_1}(w_*))$.*

Proof. Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$ and $|\sigma_2^*| = n_2$, with vertex set $\{s_1, s_2, \cdots, s_{n_1}\}$ and $\{r_1, r_2, \cdots, r_{n_2}\}$ respectively. Let s_i and r_j be two arbitrary vertices from G_1 and G_2, respectively. Let $(G_1.G_2)(s_i, r_j : w_*)$ with the vertex set $\{w_1, w_2, \cdots, w_{n_1+n_2-1}\}$ be the fuzzy graph coalescence of G_1 and G_2 at vertices s_i and r_j formed by identifying these vertices to a new vertex called w_*. Then the first K-Banhatti index is,

$$\begin{aligned}\mathcal{B}_1(G_1.G_2)(s_i, r_j : w_*) &= \sum_{ew} [d_{G_1.G_2}(e) + d_{G_1.G_2}(w)] \\ &= \sum_{i=1}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) + d_{G_1.G_2}(w_i)]\end{aligned}$$

The equation can be executed by categorizing the vertices into three distinct cases based on their characteristics.

Case 1: Consider those vertices which are different from w_* and the vertices belong to the neighborhood of w_*.

$$\sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*)}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) + d_{G_1.G_2}(w_i)]$$

$$= \sum_{\substack{i=1\\ i\neq *\\ w_i\notin N(w_*)\\ w_i\in G_1}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_1}(w_i w_j)+d_{G_1}(w_i)] + \sum_{\substack{i=1\\ i\neq *\\ w_i\notin N(w_*)\\ w_i\in G_2}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_2}(w_i w_j)+d_{G_2}(w_i)]$$

Case 2: Consider the vertex w_*.
Here it can again be considered as edge in G_1 and G_2.

$$\begin{aligned}
&\sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j)+d_{G_1.G_2}(w_*)]\\
&\quad= \sum_{\substack{j=1\\ j\in G_1}}^{m_i}[d_{G_1.G_2}(w_* w_j)+d_{G_1.G_2}(w_*)] + \sum_{\substack{j=1\\ j\in G_2}}^{m_i}[d_{G_1.G_2}(w_* w_j)+d_{G_1.G_2}(w_*)]\\
&\quad= \sum_{\substack{j=1\\ j\in G_1}}^{m_i}[d_{G_1}(w_* w_j)+d_{G_2}(w_*)+d_{G_1}(w_*)+d_{G_2}(w_*)]\\
&\qquad+ \sum_{\substack{j=1\\ j\in G_2}}^{m_i}[d_{G_2}(w_* w_j)+d_{G_1}(w_*)+d_{G_1}(w_*)+d_{G_2}(w_*)]
\end{aligned}$$

Case 3: Consider all those vertices which belongs to the neighborhood of w_*.
Again it can be divided as edge in G_1 and G_2

$$\begin{aligned}
&\sum_{w_i\in N(w_*)} \sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j)+d_{G_1.G_2}(w_i)]\\
&\quad= \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}} \sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1.G_2}(w_i w_j)+d_{G_1.G_2}(w_i)]\\
&\qquad+ \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}} [d_{G_1.G_2}(w_i w_*)+d_{G_1.G_2}(w_i)]\\
&\qquad+ \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}} \sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1.G_2}(w_i w_j)+d_{G_1.G_2}(w_i)]\\
&\qquad+ \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}} [d_{G_1.G_2}(w_i w_*)+d_{G_1.G_2}(w_i)]
\end{aligned}$$

$$= \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_1}(w_i w_j) + d_{G_1}(w_i)]$$
$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [d_{G_1}(w_i w_*) + d_{G_2}(w_*) + d_{G_1}(w_i)]$$
$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_2}(w_i w_j) + d_{G_2}(w_i)]$$
$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [d_{G_2}(w_i w_*) + d_{G_1}(w_*) + d_{G_2}(w_i)]$$

From cases (1), (2), and (3) it can be concluded that

$$\mathcal{B}_1(G_1.G_2)(s_i, r_j : w_*)$$
$$= \sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*)}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) + d_{G_1.G_2}(w_i)] + \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) + d_{G_1.G_2}(w_*)]$$
$$+ \sum_{w_i \in N(w_*)} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) + d_{G_1.G_2}(w_i)]$$
$$= \sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*) \\ w_i \in G_1}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1}(w_i w_j) + d_{G_1}(w_i)] + \sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*) \\ w_i \in G_2}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_2}(w_i w_j) + d_{G_2}(w_i)]$$
$$+ \sum_{\substack{j=1 \\ j \in G_1}}^{m_i} [d_{G_1}(w_* w_j) + d_{G_2}(w_*) + d_{G_1}(w_*) + d_{G_2}(w_*)]$$
$$+ \sum_{\substack{j=1 \\ j \in G_2}}^{m_i} [d_{G_2}(w_* w_j) + d_{G_1}(w_*) + d_{G_1}(w_*) + d_{G_2}(w_*)]$$
$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_1}(w_i w_j) + d_{G_1}(w_i)]$$
$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [d_{G_1}(w_i w_*) + d_{G_2}(w_*) + d_{G_1}(w_i)]$$

$$
\begin{aligned}
&+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}\sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_2}(w_iw_j)+d_{G_2}(w_i)]\\
&+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}[d_{G_2}(w_iw_*)+d_{G_1}(w_*)+d_{G_2}(w_i)]\\
&=\mathcal{B}_1(G_1)+\mathcal{B}_1(G_2)+2\sum_{\substack{j=1\\ j\in G_1}}^{m_i}d_{G_2}(w_*)+2\sum_{\substack{j=1\\ j\in G_2}}^{m_i}d_{G_1}(w_*)\\
&+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}d_{G_2}(w_*)+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}d_{G_1}(w_*)\\
&=\mathcal{B}_1(G_1)+\mathcal{B}_1(G_2)+3(deg_{G_1}(w_*)d_{G_2}(w_*)+deg_{G_2}(w_*)d_{G_1}(w_*)).
\end{aligned}
$$

■

Corollary 13.2.2. *Let C_1 and C_2 be two fuzzy cycles with $|\sigma_1^*|=n_1$ and $|\sigma_2^*|=n_2$. Let $\{s_1,s_2,\cdots,s_{n_1}\}$ be the vertex set of C_1 and $\{r_1,r_2,\cdots,r_{n_2}\}$ be the vertex set of C_2. Let s_i and r_j be arbitrary vertices from C_1 and C_2. Then the first K-Banhatti index of fuzzy graph coalescence of C_1 and C_2 at s_i and r_j, $\mathcal{B}_1(C_1.C_2)(s_i,r_j:w_*)=\mathcal{B}_1(C_1)+\mathcal{B}_1(C_2)+6(d_{G_1}(w_*)+d_{G_2}(w_*))$.*

Theorem 13.2.3. *Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*|=n_1$ and $|\sigma_2^*|=n_2$. Let $\{s_1,s_2,\cdots,s_{n_1}\}$ be the vertex set of G_1 and $\{r_1,r_2,\cdots,r_{n_2}\}$ be the vertex set of G_2. Let s_i and r_j be arbitrary vertices from G_1 and G_2. Then the second K-Banhatti index of fuzzy graph coalescence of G_1 and G_2 at s_i and r_j,*

$$
\begin{aligned}
&\mathcal{B}_2(G_1.G_2)(s_i,r_j:w_*)\\
&=\mathcal{B}_2(G_1)+\mathcal{B}_2(G_2)+[deg_{G_1}(w_*)+deg_{G_2}(w_*)][d_{G_1}(w_*)\times d_{G_2}(w_*)]\\
&+d_{G_1}(w_*)\left[\sum_{\substack{j=1\\ j\in G_2}}^{m_i}[d_{G_2}(w_*w_j)]+deg_{G_2}(w_*)d_{G_1}(w_*)+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}[d_{G_2}(w_i)]\right]\\
&+d_{G_2}(w_*)\left[\sum_{\substack{j=1\\ j\in G_1}}^{m_i}[d_{G_1}(w_*w_j)]+deg_{G_1}(w_*)d_{G_2}(w_*)+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}[d_{G_1}(w_i)]\right].
\end{aligned}
$$

Proof. Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*|=n_1$ and $|\sigma_2^*|=n_2$, with vertex set $\{s_1,s_2,\cdots,s_{n_1}\}$ and $\{r_1,r_2,\cdots,r_{n_2}\}$ respectively. Let s_i and r_j be two arbitrary vertices from G_1 and G_2 respectively. Let $(G_1.G_2)(s_i,r_j:w_*)$ with the vertex set $\{w_1,w_2,\cdots,w_{n_1+n_2-1}\}$ be the fuzzy graph coalescence of G_1 and G_2 at vertices s_i and r_j formed by identifying these vertices to a new vertex called w_*. Then the second K-Banhatti index is,

$$\mathcal{B}_2(G_1.G_2)(s_i, r_j : w_*) = \sum_{ew}[d_{G_1.G_2}(e) \times d_{G_1.G_2}(w)]$$
$$= \sum_{i=1}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$

The equation can be executed by categorizing the vertices into three distinct cases based on their characteristics.

Case 1. Consider those vertices which are different from w_* and the vertices belongs to the neighborhood of w_*.

$$\sum_{\substack{i=1 \\ i\neq * \\ w_i \notin N(w_*)}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$
$$= \sum_{\substack{i=1 \\ i\neq * \\ w_i \notin N(w_*) \\ w_i \in G_1}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_1}(w_i w_j) \times d_{G_1}(w_i)] + \sum_{\substack{i=1 \\ i\neq * \\ w_i \notin N(w_*) \\ w_i \in G_2}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_2}(w_i w_j) \times d_{G_2}(w_i)]$$

Case 2. Consider the vertex w_*.
Here it can again be considered as edge in G_1 and G_2.

$$\sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_*)]$$
$$= \sum_{\substack{j=1 \\ j\in G_1}}^{m_i} [d_{G_1.G_2}(w_* w_j) \times d_{G_1.G_2}(w_*)] + \sum_{\substack{j=1 \\ j\in G_2}}^{m_i} [d_{G_1.G_2}(w_* w_j) \times d_{G_1.G_2}(w_*)]$$
$$= \sum_{\substack{j=1 \\ j\in G_1}}^{m_i} [(d_{G_1}(w_* w_j) + d_{G_2}(w_*)) \times (d_{G_1}(w_*) + d_{G_2}(w_*))]$$
$$+ \sum_{\substack{j=1 \\ j\in G_2}}^{m_i} [(d_{G_2}(w_* w_j) + d_{G_1}(w_*)) \times (d_{G_1}(w_*) + d_{G_2}(w_*))]$$
$$= \sum_{\substack{j=1 \\ j\in G_1}}^{m_i} [(d_{G_1}(w_* w_j) \times (d_{G_1}(w_*)) + ((d_{G_1}(w_* w_j) \times (d_{G_2}(w_*))$$
$$+ (d_{G_1}(w_*) \times d_{G_2}(w_*)) + d_{G_2}^2(w_*)]$$

$$+\sum_{\substack{j=1\\ j\in G_2}}^{m_i}[(d_{G_2}(w_*w_j)\times d_{G_1}(w_*))+(d_{G_2}(w_*w_j)\times d_{G_2}(w_*))$$
$$+d_{G_1}^2(w_*)+(d_{G_1}(w_*)\times d_{G_2}(w_*))]$$

Case 3. Consider all those vertices which belongs to the neighborhood of w_*. Again it can be divided as edge in G_1 and G_2

$$\sum_{w_i\in N(w_*)}\sum_{j=1}^{m_i}[d_{G_1.G_2}(w_iw_j)\times d_{G_1.G_2}(w_i)]$$
$$=\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}\sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1.G_2}(w_iw_j)\times d_{G_1.G_2}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}[d_{G_1.G_2}(w_iw_*)\times d_{G_1.G_2}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}\sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1.G_2}(w_iw_j)\times d_{G_1.G_2}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}[d_{G_1.G_2}(w_iw_*)\times d_{G_1.G_2}(w_i)]$$
$$=\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}\sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1}(w_iw_j)\times d_{G_1}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}[(d_{G_1}(w_iw_*)+d_{G_2}(w_*))\times d_{G_1}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}\sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_2}(w_iw_j)\times d_{G_2}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}[(d_{G_2}(w_iw_*)+d_{G_1}(w_*))\times d_{G_2}(w_i)]$$
$$=\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}\sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1}(w_iw_j)\times d_{G_1}(w_i)]$$
$$+\sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}[(d_{G_1}(w_iw_*)\times d_{G_1}(w_i))+(d_{G_2}(w_*)\times d_{G_1}(w_i))]$$

$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_2}(w_i w_j) \times d_{G_2}(w_i)]$$

$$+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [(d_{G_2}(w_i w_*) \times d_{G_2}(w_i)) + (d_{G_1}(w_*) \times d_{G_2}(w_i))]$$

From cases (1), (2), and (3) it can be concluded that

$$\begin{aligned}
&\mathcal{B}_2(G_1.G_2)(s_i, r_j : w_*) \\
&= \sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*)}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)] + \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_*)] \\
&+ \sum_{w_i \in N(w_*)} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)] \\
&= \sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*) \\ w_i \in G_1}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1}(w_i w_j) \times d_{G_1}(w_i)] + \sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*) \\ w_i \in G_2}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_2}(w_i w_j) \times d_{G_2}(w_i)] \\
&+ \sum_{\substack{j=1 \\ j \in G_1}}^{m_i} [(d_{G_1}(w_* w_j) \times d_{G_1}(w_*)) + (d_{G_1}(w_* w_j) \times d_{G_2}(w_*)) \\
&\quad + (d_{G_1}(w_*) \times d_{G_2}(w_*)) + d_{G_2}^2(w_*)] \\
&+ \sum_{\substack{j=1 \\ j \in G_2}}^{m_i} [(d_{G_2}(w_* w_j) \times d_{G_1}(w_*)) + (d_{G_2}(w_* w_j) \times d_{G_2}(w_*)) \\
&\quad + d_{G_1}^2(w_*) + (d_{G_1}(w_*) \times d_{G_2}(w_*))] \\
&+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_1}(w_i w_j) \times d_{G_1}(w_i)] \\
&+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [(d_{G_1}(w_i w_*) \times d_{G_1}(w_i)) + (d_{G_2}(w_*) \times d_{G_1}(w_i))] \\
&+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_2}(w_i w_j) \times d_{G_2}(w_i)]
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [(d_{G_2}(w_i w_*) \times d_{G_2}(w_i)) + (d_{G_1}(w_*) \times d_{G_2}(w_i))] \\
&= \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) \\
&+ \sum_{\substack{j=1 \\ j \in G_1}}^{m_i} [(d_{G_1}(w_* w_j) \times d_{G_2}(w_*)) + (d_{G_1}(w_*) \times d_{G_2}(w_*)) + d_{G_2}^2(w_*)] \\
&+ \sum_{\substack{j=1 \\ j \in G_2}}^{m_i} [(d_{G_2}(w_* w_j) \times d_{G_1}(w_*)) + d_{G_1}^2(w_*) + (d_{G_1}(w_*) \times d_{G_2}(w_*))] \\
&+ \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [(d_{G_2}(w_*) \times d_{G_1}(w_i))] + \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [(d_{G_1}(w_*) \times d_{G_2}(w_i))] \\
&= \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) + d_{G_2}(w_*) \sum_{\substack{j=1 \\ j \in G_1}}^{m_i} [d_{G_1}(w_* w_j)] \\
&+ [deg_{G_1}(w_*) + deg_{G_2}(w_*)][(d_{G_1}(w_*) \times d_{G_2}(w_*))] \\
&+ deg_{G_1}(w_*) d_{G_2}^2(w_*) + d_{G_1}(w_*) \sum_{\substack{j=1 \\ j \in G_2}}^{m_i} [d_{G_2}(w_* w_j)] + deg_{G_2}(w_*) d_{G_1}^2(w_*) \\
&+ d_{G_2}(w_*) \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [d_{G_1}(w_i)] + d_{G_1}(w_*) \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [d_{G_2}(w_i)] \\
&= \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) + [deg_{G_1}(w_*) + deg_{G_2}(w_*)][(d_{G_1}(w_*) \times d_{G_2}(w_*))] \\
&+ d_{G_1}(w_*) \left[\sum_{\substack{j=1 \\ j \in G_2}}^{m_i} [d_{G_2}(w_* w_j)] + deg_{G_2}(w_*) d_{G_1}(w_*) + \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [d_{G_2}(w_i)] \right] \\
&+ d_{G_2}(w_*) \left[\sum_{\substack{j=1 \\ j \in G_1}}^{m_i} [d_{G_1}(w_* w_j)] + deg_{G_1}(w_*) d_{G_2}(w_*) + \sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [d_{G_1}(w_i)] \right].
\end{aligned}
$$

■

Corollary 13.2.4. *Let C_1 and C_2 be two fuzzy cycles with $|\sigma_1^*| = n_1$ and $|\sigma_2^*| = n_2$. Let $\{s_1, s_2, \cdots, s_{n_1}\}$ be the vertex set of C_1 and $\{r_1, r_2, \cdots, r_{n_2}\}$ be the vertex set of C_2. Let s_i and r_j be arbitrary vertices from C_1 and C_2. Then the first K-Banhatti index of fuzzy graph coalescence of C_1 and C_2 at s_i and r_j, $\mathcal{B}_1(C_1.C_2)(s_i, r_j : w_*) = \mathcal{B}_1(C_1) + \mathcal{B}_1(C_2) + 6(d_{G_1}(w_*) + d_{G_2}(w_*))$.*

Theorem 13.2.5. *Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$ and $|\sigma_2^*| = n_2$. Let $\{s_1, s_2, \cdots, s_{n_1}\}$ be the vertex set of G_1 and $\{r_1, r_2, \cdots, r_{n_2}\}$ be the vertex set of G_2. Let s_i and r_j be arbitrary vertices from G_1 and G_2. Then the second K-Banhatti index of fuzzy graph coalescence of G_1 and G_2 at s_i and r_j,*

$$\begin{aligned}
&\mathcal{B}_2(G_1.G_2)(s_i, r_j : w_*)\\
&\quad = \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) + [deg_{G_1}(w_*) + deg_{G_2}(w_*)][d_{G_1}(w_*) \times d_{G_2}(w_*)]\\
&\qquad + d_{G_1}(w_*)\left[\sum_{\substack{j=1\\ j\in G_2}}^{m_i}[d_{G_2}(w_* w_j)] + deg_{G_2}(w_*) d_{G_1}(w_*) + \sum_{\substack{w_i \in N(w_*)\\ w_i \in G_2}}[d_{G_2}(w_i)]\right]\\
&\qquad + d_{G_2}(w_*)\left[\sum_{\substack{j=1\\ j\in G_1}}^{m_i}[d_{G_1}(w_* w_j)] + deg_{G_1}(w_*) d_{G_2}(w_*) + \sum_{\substack{w_i \in N(w_*)\\ w_i \in G_1}}[d_{G_1}(w_i)]\right].
\end{aligned}$$

Proof. Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$ and $|\sigma_2^*| = n_2$, with vertex set $\{s_1, s_2, \cdots, s_{n_1}\}$ and $\{r_1, r_2, \cdots, r_{n_2}\}$ respectively. Let s_i and r_j be two arbitrary vertices from G_1 and G_2 respectively. Let $(G_1.G_2)(s_i, r_j : w_*)$ with the vertex set $\{w_1, w_2, \cdots, w_{n_1+n_2-1}\}$ be the fuzzy graph coalescence of G_1 and G_2 at vertices s_i and r_j formed by identifying these vertices to a new vertex called w_*. Then the second K-Banhatti index is,

$$\begin{aligned}
\mathcal{B}_2(G_1.G_2)(s_i, r_j : w_*) &= \sum_{ew}[d_{G_1.G_2}(e) \times d_{G_1.G_2}(w)]\\
&= \sum_{i=1}^{n_1+n_2-1}\sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]
\end{aligned}$$

The equation can be executed by categorizing the vertices into three distinct cases based on their characteristics.

Case 1: Consider those vertices which are different from w_* and the vertices belongs to the neighborhood of w_*.

$$\begin{aligned}
&\sum_{\substack{i=1\\ i\neq *\\ w_i \notin N(w_*)}}^{n_1+n_2-1}\sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]\\
&= \sum_{\substack{i=1\\ i\neq *\\ w_i \notin N(w_*)\\ w_i \in G_1}}^{n_1+n_2-1}\sum_{j=1}^{m_i}[d_{G_1}(w_i w_j) \times d_{G_1}(w_i)] + \sum_{\substack{i=1\\ i\neq *\\ w_i \notin N(w_*)\\ w_i \in G_2}}^{n_1+n_2-1}\sum_{j=1}^{m_i}[d_{G_2}(w_i w_j) \times d_{G_2}(w_i)]
\end{aligned}$$

Case 2: Consider the vertex w_*.
Here it can again be considered as edge in G_1 and G_2.

$$\sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_*)]$$

$$= \sum_{\substack{j=1 \\ j\in G_1}}^{m_i} [d_{G_1.G_2}(w_* w_j) \times d_{G_1.G_2}(w_*)] + \sum_{\substack{j=1 \\ j\in G_2}}^{m_i} [d_{G_1.G_2}(w_* w_j) \times d_{G_1.G_2}(w_*)]$$

$$= \sum_{\substack{j=1 \\ j\in G_1}}^{m_i} [(d_{G_1}(w_* w_j) + d_{G_2}(w_*)) \times (d_{G_1}(w_*) + d_{G_2}(w_*))]$$

$$+ \sum_{\substack{j=1 \\ j\in G_2}}^{m_i} [(d_{G_2}(w_* w_j) + d_{G_1}(w_*)) \times (d_{G_1}(w_*) + d_{G_2}(w_*))]$$

$$= \sum_{\substack{j=1 \\ j\in G_1}}^{m_i} [(d_{G_1}(w_* w_j) \times (d_{G_1}(w_*)) + ((d_{G_1}(w_* w_j) \times (d_{G_2}(w_*))$$

$$+ (d_{G_1}(w_*) \times d_{G_2}(w_*)) + d^2_{G_2}(w_*)]$$

$$+ \sum_{\substack{j=1 \\ j\in G_2}}^{m_i} [(d_{G_2}(w_* w_j) \times d_{G_1}(w_*)) + (d_{G_2}(w_* w_j) \times d_{G_2}(w_*))$$

$$+ d^2_{G_1}(w_*) + (d_{G_1}(w_*) \times d_{G_2}(w_*))]$$

Case 3: Consider all those vertices which belongs to the neighborhood of w_*.
Again it can be divided as edge in G_1 and G_2

$$\sum_{w_i\in N(w_*)} \sum_{j=1}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$

$$= \sum_{\substack{w_i\in N(w_*) \\ w_i\in G_1}} \sum_{\substack{j=1 \\ j\neq *}}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$

$$+ \sum_{\substack{w_i\in N(w_*) \\ w_i\in G_1}} [d_{G_1.G_2}(w_i w_*) \times d_{G_1.G_2}(w_i)]$$

$$+ \sum_{\substack{w_i\in N(w_*) \\ w_i\in G_2}} \sum_{\substack{j=1 \\ j\neq *}}^{m_i}[d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [d_{G_1.G_2}(w_i w_*) \times d_{G_1.G_2}(w_i)]$$

$$=\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_1}(w_i w_j) \times d_{G_1}(w_i)]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [(d_{G_1}(w_i w_*) + d_{G_2}(w_*)) \times d_{G_1}(w_i)]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_2}(w_i w_j) \times d_{G_2}(w_i)]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [(d_{G_2}(w_i w_*) + d_{G_1}(w_*)) \times d_{G_2}(w_i)]$$

$$=\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_1}(w_i w_j) \times d_{G_1}(w_i)]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_1}} [(d_{G_1}(w_i w_*) \times d_{G_1}(w_i)) + (d_{G_2}(w_*) \times d_{G_1}(w_i))]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} \sum_{\substack{j=1 \\ j \neq *}}^{m_i} [d_{G_2}(w_i w_j) \times d_{G_2}(w_i)]$$

$$+\sum_{\substack{w_i \in N(w_*) \\ w_i \in G_2}} [(d_{G_2}(w_i w_*) \times d_{G_2}(w_i)) + (d_{G_1}(w_*) \times d_{G_2}(w_i))]$$

From cases (1), (2), and (3) it can be concluded that

$$\mathcal{B}_2(G_1.G_2)(s_i, r_j : w_*)$$
$$=\sum_{\substack{i=1 \\ i \neq * \\ w_i \notin N(w_*)}}^{n_1+n_2-1} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$
$$+\sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_*)] + \sum_{w_i \in N(w_*)} \sum_{j=1}^{m_i} [d_{G_1.G_2}(w_i w_j) \times d_{G_1.G_2}(w_i)]$$

$$
\begin{aligned}
&= \sum_{\substack{i=1\\ i\neq *\\ w_i\notin N(w_*)\\ w_i\in G_1}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_1}(w_iw_j)\times d_{G_1}(w_i)] + \sum_{\substack{i=1\\ i\neq *\\ w_i\notin N(w_*)\\ w_i\in G_2}}^{n_1+n_2-1} \sum_{j=1}^{m_i}[d_{G_2}(w_iw_j)\times d_{G_2}(w_i)]\\
&\quad + \sum_{\substack{j=1\\ j\in G_1}}^{m_i}[(d_{G_1}(w_*w_j)\times d_{G_1}(w_*)) + (d_{G_1}(w_*w_j)\times d_{G_2}(w_*))\\
&\quad + (d_{G_1}(w_*)\times d_{G_2}(w_*)) + d^2_{G_2}(w_*)]\\
&\quad + \sum_{\substack{j=1\\ j\in G_2}}^{m_i}[(d_{G_2}(w_*w_j)\times d_{G_1}(w_*)) + (d_{G_2}(w_*w_j)\times d_{G_2}(w_*))\\
&\quad + d^2_{G_1}(w_*) + (d_{G_1}(w_*)\times d_{G_2}(w_*))] + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}} \sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_1}(w_iw_j)\times d_{G_1}(w_i)]\\
&\quad + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}[(d_{G_1}(w_iw_*)\times d_{G_1}(w_i)) + (d_{G_2}(w_*)\times d_{G_1}(w_i))]\\
&\quad + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}} \sum_{\substack{j=1\\ j\neq *}}^{m_i}[d_{G_2}(w_iw_j)\times d_{G_2}(w_i)]\\
&\quad + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}[(d_{G_2}(w_iw_*)\times d_{G_2}(w_i)) + (d_{G_1}(w_*)\times d_{G_2}(w_i))]\\
&= \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) + \sum_{\substack{j=1\\ j\in G_1}}^{m_i}[(d_{G_1}(w_*w_j)\times d_{G_2}(w_*))\\
&\quad + (d_{G_1}(w_*)\times d_{G_2}(w_*)) + d^2_{G_2}(w_*)]\\
&\quad + \sum_{\substack{j=1\\ j\in G_2}}^{m_i}[(d_{G_2}(w_*w_j)\times d_{G_1}(w_*)) + d^2_{G_1}(w_*) + (d_{G_1}(w_*)\times d_{G_2}(w_*))]\\
&\quad + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}}[(d_{G_2}(w_*)\times d_{G_1}(w_i))] + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}}[(d_{G_1}(w_*)\times d_{G_2}(w_i))]\\
&= \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) + d_{G_2}(w_*) \sum_{\substack{j=1\\ j\in G_1}}^{m_i}[d_{G_1}(w_*w_j)]\\
&\quad + [deg_{G_1}(w_*) + deg_{G_2}(w_*)][(d_{G_1}(w_*)\times d_{G_2}(w_*))]
\end{aligned}
$$

$$+ deg_{G_1}(w_*)d^2_{G_2}(w_*) + d_{G_1}(w_*) \sum_{\substack{j=1\\ j\in G_2}}^{m_i} [d_{G_2}(w_* w_j)] + deg_{G_2}(w_*)d^2_{G_1}(w_*)$$

$$+ d_{G_2}(w_*) \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}} [d_{G_1}(w_i)] + d_{G_1}(w_*) \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}} [d_{G_2}(w_i)]$$

$$= \mathcal{B}_2(G_1) + \mathcal{B}_2(G_2) + [deg_{G_1}(w_*) + deg_{G_2}(w_*)][(d_{G_1}(w_*) \times d_{G_2}(w_*))]$$

$$+ d_{G_1}(w_*) \left[\sum_{\substack{j=1\\ j\in G_2}}^{m_i} [d_{G_2}(w_* w_j)] + deg_{G_2}(w_*)d_{G_1}(w_*) + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_2}} [d_{G_2}(w_i)] \right]$$

$$+ d_{G_2}(w_*) \left[\sum_{\substack{j=1\\ j\in G_1}}^{m_i} [d_{G_1}(w_* w_j)] + deg_{G_1}(w_*)d_{G_2}(w_*) + \sum_{\substack{w_i\in N(w_*)\\ w_i\in G_1}} [d_{G_1}(w_i)] \right].$$

■

Example 13.2.6. Consider the fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ given in Fig. 13.2 with $\sigma_1^* = \{s_1, s_2, s_3, s_4\}$ and $\sigma_2^* = \{r_1, r_2, r_3\}$ where $\mu_1(s_1s_2) = 0.3$, $\mu_1(s_1s_4) = 0.4$, $\mu_1(s_2s_3) = 0.2$, $\mu_1(s_2s_4) = 0.5$, $\mu_1(s_3s_4) = 0.7$, $\mu_2(r_1r_2) = 0.3$, $\mu_2(r_1r_3) = 0.2$, and $\mu_2(r_2r_3) = 0.9$.

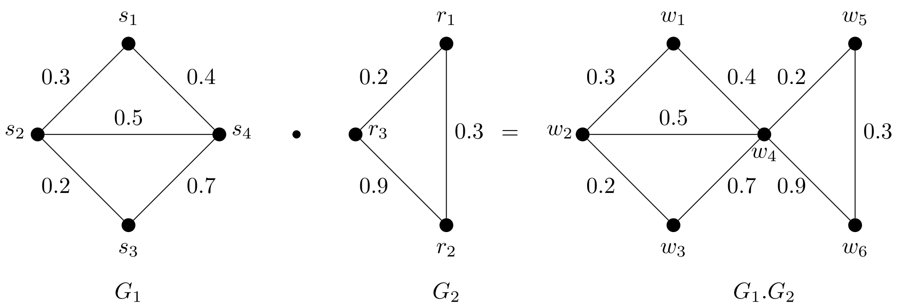

FIGURE 13.2

Fuzzy coalescence of two fuzzy graphs G_1 and G_2.

After finding coalescence of G_1 and G_2, we calculate $\mathcal{B}_1(G_1.G_2)(s_4, r_3 : w_*) = 1.1 + 0.7 + 1.1 + 1 + 1.5 + 1 + 1.5 + 0.9 + 2.6 + 0.7 + 2.6 + 2.7 + 2.2 + 0.9 + 2.2 + 2.7 + 1.1 + 1.2 + 1.1 + 0.5 + 2.8 + 2.7 + 2.8 + 0.5 + 2.1 + 2.7 + 2.1 + 1.2 + 2.7 + 1 + 2.7 + 2.7 = 55.3$. Now using Theorem 13.2.1 we can find this without actually finding $(G_1.G_2)(s_4, r_3 : w_*)$, $\mathcal{B}_1(G_1.G_2)(s_4, r_3 : w_*) = 1.1 + 0.7 + 1.1 + 1 + 1.5 + 1 + 1.5 + 0.9 + 1.5 + 0.7 + 1.5 + 1.6 + 1.1 + 0.9 + 1.1 + 1.6 + 1.6 + 1 + 1.6 + 1.6 + 8(0.2 + 0.3 + 0.9) + 3(3 \times 1.1 + 2 \times 1.6) = 55.3$.

Similarly after finding coalescence of G_1 and G_2, we calculate $\mathcal{B}_2(G_1.G_2)(s_4, r_3 : w_*) = 1.1 \times 0.7 + 1.1 \times 1 + 1.5 \times 1 + 1.5 \times 0.9 + 2.6 \times 0.7 + 2.6 \times 2.7 + 2.2 \times 0.9 + 2.2 \times 2.7 + 1.1 \times 1.2 + 1.1 \times 0.5 + 2.8 \times 2.7 + 2.8 \times 0.5 + 2.1 \times 2.7 + 2.1 \times 1.2 + 2.7 \times 1 + 2.7 \times 2.7 = 50.49$. Now using Theorem 13.2.5, we can find this without actually finding $(G_1.G_2)(s_4, r_3 : w_*)$, $\mathcal{B}_2(G_1.G_2)(s_4, r_3 : w_*) = 1.1 \times 0.7 + 1.1 \times 1 + 1.5 \times 1 + 1.5 \times 0.9 + 1.5 \times 0.7 + 1.5 \times 1.6 + 1.1 \times 0.9 + 1.1 \times 1.6 + 1.6 \times 1 + 1.6 \times 1.6 + 1.1 \times 1.2 + 1.1 \times 0.5 + 1.2 \times 1.1 + 1.2 \times 0.5 + 0.5 \times 1.1 + 0.5 \times 1.2 + ([3 + 2] \times 1.6 \times 1.1) + 1.6 \times ((1.2 + 0.5) + (2 \times 1.6) + (0.5 + 1.2)) + 1.1((1.5 + 1.6 + 1.1) + (3 \times 1.1) + (0.7 + 1 + 0.9)) = 50.49$

Theorem 13.2.7. *Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$, $|\sigma_2^*| = n_2$ and $|\mu_1^*| = m_1$, $|\mu_2^*| = m_2$. Suppose the weight of all vertices in G_1 and G_2 is r, then the first K-Banhatti index of corona product of fuzzy graphs G_1 and G_2 is $\mathcal{B}_1(G_1 \circ G_2) = B_1(G_1) + n_1 B_1(G_2) + 6(Tn_2 + Sn_1) + 6r(m_1n_2 + m_2n_1) + rn_1n_2[3n_2 - 1]$, where T is the sum of all weights of edges in the fuzzy graph G_1 and S is the sum of all weight of edges in the fuzzy graph G_2.*

Proof. Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$, $|\sigma_2^*| = n_2$, and $|\mu_1^*| = m_1$, $|\mu_2^*| = m_2$. Suppose $\sigma(w_i) = r$ for all vertices in G_1 and G_2, T be the sum of all weight of edges in the fuzzy graph G_1 and S be the sum of all weights of edges in the fuzzy graph G_2. Then $\mathcal{B}_1(G_1 \circ G_2) = \sum_{ew}[d_{G_1 \circ G_2}(e) + d_{G_1 \circ G_2}(w)] = \sum_{i=1}^{n_1(1+n_2)} \sum_{j=1}^{m_i}[d_{G_1 \circ G_2}(w_iw_j) + d_{G_1 \circ G_2}(w_i)]$. Here the vertex set of $G_1 \circ G_2$ can be treated as two cases. The first case is those vertices that belongs to G_1 and the second case is those vertices which belongs to the replica of G_2. Now let us call the union of replicas of G_2 as G_2'. Now consider case one, it can again be divided as two subcases, first one is those edges fully lies in G_1 itself. Second subcase is those edges having one vertex in G_1 and other in G_2'. For subcase one $\sum_{w_i \in G_1} \sum_{j=1}^{m_i}[d_{G_1 \circ G_2}(w_iw_j) + d_{G_1 \circ G_2}(w_i)] = \sum_{w_i \in G_1} \sum_{j=1}^{m_i}[(d_{G_1}(w_iw_j) + 2n_2r) + (d_{G_1}(w_i) + n_2r)] = \sum_{w_i \in G_1} \sum_{j=1}^{m_i}[d_{G_1}(w_iw_j) + d_{G_1}(w_i) + 3n_2r]$. And for subcase two $\sum_{w_i \in G_1} \sum_{j=1}^{m_i}[d_{G_1 \circ G_2}(w_iw_j) + d_{G_1 \circ G_2}(w_i)] = \sum_{w_i \in G_1} \sum_{j=1}^{m_i}[(d_{G_1}(w_i) + d_{G_2}(w_j) + (n_2 - 1)r) + (d_{G_1}(w_i) + n_2r)] = \sum_{w_i \in G_1} \sum_{j=1}^{m_i}[2d_{G_1}(w_i) + d_{G_2}(w_j) + (2n_2 - 1)r]$. Now consider case two, here also it has two subcases as of case one. The first one consist of edges that fully lie in G_2' itself. Second subcase is those edges having one vertex in G_2' and other in G_1. For subcase one $\sum_{w_i \in G_2'} \sum_{j=1}^{m_i}[d_{G_1 \circ G_2}(w_iw_j) + d_{G_1 \circ G_2}(w_i)] =$

$$\sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [(d_{G_2}(w_i w_j) + 2r) + (d_{G_2}(w_i) + r)] = \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_2}(w_i w_j) + d_{G_2}(w_i) + 3r]$$

and for subcase two $\sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_1 \circ G_2}(w_i w_j) + d_{G_1 \circ G_2}(w_i)] = \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [(d_{G_2}(w_i) + d_{G_1}(w_j) + (n_2 - 1)r) + (d_{G_2}(w_i) + r)] = \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_1}(w_j) + 2d_{G_2}(w_i) + n_2 r]$.

Now from cases 1 and 2 and its subcases, it can be concluded that

$$\mathcal{B}_1(G_1 \circ G_2) = \sum_{w_i \in G_1} \sum_{j=1}^{m_i} [d_{G_1}(w_i w_j) + d_{G_1}(w_i) + 3n_2 r] + \sum_{w_i \in G_1} \sum_{j=1}^{m_i} [2d_{G_1}(w_i) + d_{G_2}(w_j) + (2n_2 - 1)r]$$
$$+ \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_2}(w_i w_j) + d_{G_2}(w_i) + 3r] + \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_1}(w_j) + 2d_{G_2}(w_i) + n_2 r]$$
$$= B_1(G_1) + n_1 B_1(G_2) + 3 \sum_{w_i \in G_1} \sum_{j=1}^{m_i} n_2 r + \sum_{w_i \in G_1} \sum_{j=1}^{m_i} [2d_{G_1}(w_i) + d_{G_2}(w_j) + (2n_2 - 1)r] + 3 \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} r + \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_1}(w_j) + 2d_{G_2}(w_i) + n_2 r] = B_1(G_1) + n_1 B_1(G_2) +$$
$$6m_1 n_2 r + 4T n_2 + 2S n_1 + n_1 n_2[(2n_2 - 1)r] + 6m_2 n_1 r + 2T n_2 + 4S n_1 + n_1 n_2^2 r$$
$$= B_1(G_1) + n_1 B_1(G_2) + 6(T n_2 + S n_1) + 6r(m_1 n_2 + m_2 n_1) + r n_1 n_2[3n_2 - 1]. \quad \blacksquare$$

Theorem 13.2.8. *Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$, $|\sigma_2^*| = n_2$, and $|\mu_1^*| = m_1$, $|\mu_2^*| = m_2$. Let $t_i^{\prime}s$ be the weight of edges in G_1 and $s_i^{\prime}s$ be the weight of edges in G_2 and the weight of all vertices in G_1 and G_2 is r, then the second K-Banhatti index of corona product of fuzzy graphs G_1 and G_2 is*

$$\mathcal{B}_2(G_1 \circ G_2) = \mathcal{B}_2(G_1) + n_1 \mathcal{B}_2(G_2) + 2r(n_1(Q + \sum_{w_i \in G_2} [deg_{G_2}(w_i) d_{G_2}(w_i)])$$
$$+ n_2(R + 2Sn_1 + 2Tn_2 + \sum_{w_i \in G_1} [deg_{G_1}(w_i) d_{G_1}(w_i)]))$$
$$+ 2(n_2[\sum_{i=1}^{n_1} t_i^2 + \sum_{d(w_i, w_j)=1} t_i t_j] + n_1[\sum_{i=1}^{n_2} s_i^2 + \sum_{d(w_i, w_j)=1} s_i s_j])$$
$$+ 8TS + r^2(4m_1 n_2^2 + 4n_1 m_2 + (n_2^2 - 1^2) n_1 n_2),$$

where T is the sum of all weight of edges in the fuzzy graph G_1, S is the sum of all weight of edges in the fuzzy graph G_2, R be the sum of degree of all edges in the fuzzy graph G_1 and Q be the sum of degree of all edges in the fuzzy graph G_2.

Proof. Let G_1 and G_2 be two fuzzy graphs with $|\sigma_1^*| = n_1$, $|\sigma_2^*| = n_2$, and $|\mu_1^*| = m_1$, $|\mu_2^*| = m_2$. Suppose that $\sigma(w_i) = r$ for all vertices in G_1 and G_2. Let $t_i's$ be the weight of edges in G_1 and $s_i's$ be the weight of edges in G_2. By definition same weights are followed in $G_1 \circ G_2$ also. Let T be the sum of all weight of edges in the fuzzy graph G_1, S be the sum of all weight of edges in the fuzzy graph G_2, R be the sum of degree of all edges in the fuzzy graph G_1 and Q be the sum of degree of all edges in the fuzzy graph G_2. Then $\mathcal{B}_2(C) = \sum_{ew}[d_{G_1\circ G_2}(e) \times d_{G_1\circ G_2}(w)] = \sum_{i=1}^{n_1(1+n_2)} \sum_{j=1}^{m_i}[d_{G_1\circ G_2}(w_i w_j) \times d_{G_1\circ G_2}(w_i)]$. Here the vertex set of $G_1 \circ G_2$ can be treated as two cases. The first case is those vertices which belongs to G_1 and the second case is those vertices which belongs to the replica of G_2. Now let us call the union of replicas of G_2 as G_2'. Now consider case one, it can again be divided as two sub cases, first one is those edges that fully lies in G_1 itself. Second subcase is those edges having one vertex in G_1 and other in G_2'. For subcase one $\sum_{w_i\in G_1} \sum_{j=1}^{m_i}[d_{G_1\circ G_2}(w_i w_j) \times d_{G_1\circ G_2}(w_i)] = \sum_{w_i\in G_1} \sum_{j=1}^{m_i}[(d_{G_1}(w_i w_j) + 2n_2 r) \times (d_{G_1}(w_i) + n_2 r)] = \sum_{w_i\in G_1} \sum_{j=1}^{m_i}[d_{G_1}(w_i w_j) d_{G_1}(w_i) + d_{G_1}(w_i w_j) n_2 r + 2n_2 r d_{G_1}(w_i) + 2n_2^2 r^2]$. For subcase two $\sum_{w_i\in G_1} \sum_{j=1}^{m_i}[d_{G_1\circ G_2}(w_i w_j) \times d_{G_1\circ G_2}(w_i)] = \sum_{w_i\in G_1} \sum_{j=1}^{m_i}[(d_{G_1}(w_i) + d_{G_2}(w_j) + (n_2 - 1)r) \times (d_{G_1}(w_i) + n_2 r)] = \sum_{w_i\in G_1} \sum_{j=1}^{m_i}[d_{G_1}^2(w_i) + d_{G_2}(w_j) d_{G_1}(w_i) + d_{G_2}(w_j) n_2 r + (2n_2 - 1) r d_{G_1}(w_i) + (n_2 - 1) n_2 r^2]$.

Now consider case two, here also it has two subcases as of case one. The first one is those edges fully lies in G_2' itself. Second subcase is those edges having one vertex in G_2' and other in G_1. For subcase one

$$\sum_{w_i\in G_2'} \sum_{j=1}^{m_i}[d_{G_1\circ G_2}(w_i w_j) \times d_{G_1\circ G_2}(w_i)] =$$

$$\sum_{w_i\in G_2'} \sum_{j=1}^{m_i}[(d_{G_2}(w_i w_j) + 2r) \times (d_{G_2}(w_i) + r)]$$

$$= \sum_{w_i\in G_2'} \sum_{j=1}^{m_i}[d_{G_2}(w_i w_j) d_{G_2}(w_i) + d_{G_2}(w_i w_j) r + 2r d_{G_2}(w_i) + 2r^2]$$ and for subcase two

$$\sum_{w_i\in G_2'} \sum_{j=1}^{m_i}[d_{G_1\circ G_2}(w_i w_j) \times d_{G_1\circ G_2}(w_i)]$$

$= \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [(d_{G_2}(w_i) + d_{G_1}(w_j) + (n_2-1)r) \times (d_{G_2}(w_i)+r)] = \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d^2_{G_2}(w_i) + d_{G_1}(w_j)d_{G_2}(w_i) + rd_{G_1}(w_j) + n_2 r d_{G_2}(w_i) + (n_2-1)r^2]$.

Now from cases 1 and 2 and its subcases, it can be concluded that $\mathcal{B}_2(G_1 \circ G_2) =$

$$\sum_{w_i \in G_1} \sum_{j=1}^{m_i} [d_{G_1}(w_i w_j) d_{G_1}(w_i) + d_{G_1}(w_i w_j) n_2 r + 2n_2 r d_{G_1}(w_i) + 2n_2^2 r^2]$$

$$+ \sum_{w_i \in G_1} \sum_{j=1}^{m_i} [d^2_{G_1}(w_i) + d_{G_2}(w_j) d_{G_1}(w_i) + d_{G_2}(w_j) n_2 r + (2n_2-1) r d_{G_1}(w_i) + (n_2 - 1) n_2 r^2]$$

$$+ \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d_{G_2}(w_i w_j) d_{G_2}(w_i) + d_{G_2}(w_i w_j) r + 2r d_{G_2}(w_i) + 2r^2]$$

$$+ \sum_{w_i \in G_2'} \sum_{j=1}^{m_i} [d^2_{G_2}(w_i) + d_{G_1}(w_j) d_{G_2}(w_i) + r d_{G_1}(w_j) + n_2 r d_{G_2}(w_i) + (n_2-1) r^2]$$

$$= \mathcal{B}_2(G_1) + n_1 \mathcal{B}_2(G_2) + 2Rn_2 r + \sum_{w_i \in G_1} [2n_2 r deg_{G_1}(w_i) d_{G_1}(w_i)] + 4m_1 n_2^2 r^2$$

$$+ 2n_2 [\sum_{i \in \mu_1 *} t_i^2 + \sum_{d(w_i,w_j)=1} t_i t_j] + 4TS + 2Sn_1 n_2 r + 2Tn_2(2n_2 - 1)r + (n_2 - 1) n_1 n_2^2 r^2$$

$$+ 2n_1 Qr + n_1 \sum_{w_i \in G_2} [2r deg_{G_2}(w_i) d_{G_2}(w_i)] + 4n_1 m_2 r^2 + 2n_1 [\sum_{i=1}^{n_2} s_i^2 + \sum_{d(w_i,w_j)=1} s_i s_j]$$

$$+ 4TS + 2Tn_2 r + 2Sn_1 n_2 r + n_1 n_2 (n_2 - 1) r^2$$

$$= \mathcal{B}_2(G_1) + n_1 \mathcal{B}_2(G_2) + 2r(n_1(Q + \sum_{w_i \in G_2} [deg_{G_2}(w_i) d_{G_2}(w_i)]) + n_2(R + 2Sn_1 + 2Tn_2 + \sum_{w_i \in G_1} [deg_{G_1}(w_i) d_{G_1}(w_i)])) + 2(n_2 [\sum_{i=1}^{n_1} t_i^2 + \sum_{d(w_i,w_j)=1} t_i t_j]$$

$$+ n_1 [\sum_{i=1}^{n_2} s_i^2 + \sum_{d(w_i,w_j)=1} s_i s_j]) + 8TS + r^2 \left(4m_1 n_2^2 + 4n_1 m_2 + (n_2^2 - 1^2) n_1 n_2\right). \quad \blacksquare$$

Example 13.2.9. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs given in Fig. 13.3 with $\sigma_1^* = \{w_1, w_2, w_3, w_4\}$ and $\sigma_2^* = \{q_1, q_2, q_3\}$. Let $\sigma_1(w_i) = 0.7$ for all i and $\sigma_2(q_i) = 0.7$ for all i. Also $\mu_1(w_1 w_2) = 0.3$, $\mu_1(w_1 w_3) = 0.5$, $\mu_1(w_1 w_4) = 0.4$, $\mu_1(w_2 w_4) = 0.2$, $\mu_1(w_3 w_4) = 0.1$, $\mu_2(q_1 q_2) = 0.4$, and $\mu_2(q_2 q_3) = 0.1$.

After finding the corona product of fuzzy graphs G_1 and G_2 as given in Fig. 13.3 we can find $\mathcal{B}_1(G_1 \circ G_2) = 5.3 + 3.3 + 5.3 + 2.6 + 5.3 + 3.3 + 5.3 + 2.8 + 5 + 3.3 + 5 + 2.7 + 5 + 2.6 + 5 + 2.8 + 5.3 + 2.8 + 5.3 + 2.7 + 1.5 + 1.1 + 1.5 + 1.2 + 1.8 + 1.2 + 1.8 + 0.8 + 1.5 + 1.1 + 1.5 + 1.2 + 1.8 + 1.2 + 1.8 + 0.8 + 1.5 + 1.1 + 1.5 + 1.2 + 1.8 +$

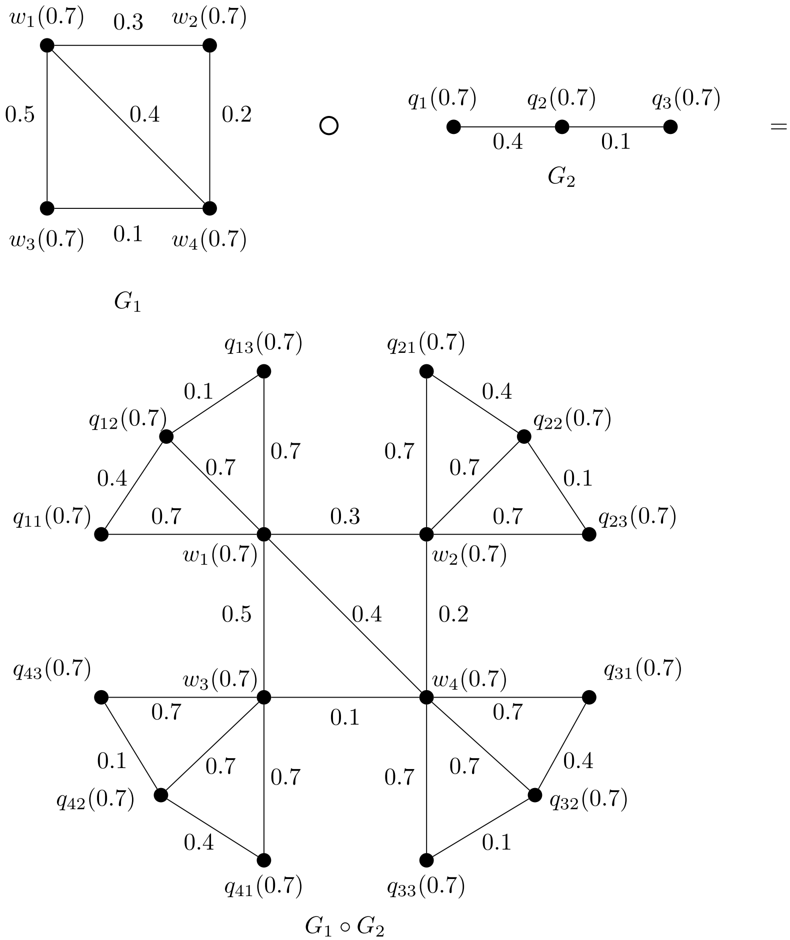

FIGURE 13.3

Corona product of two fuzzy graphs G_1 and G_2.

$1.2+1.8+0.8+1.5+1.1+1.5+1.2+1.8+1.2+1.8+0.8+3+1.1+3+3.3+3.1+1.2+3.1+3.3+2.7+0.8+2.7+3.3+2.3+1.1+2.3+2.6+2.3+1.2+2.3+2.6+2+0.8+2+2.6+2.5+1.1+2.5+2.8+2.6+1.2+2.6+2.8+2.2+0.8+2.2+2.8+2.4+1.1+2.4+2.7+2.5+1.2+2.5+2.7+2.1+0.8+2.1+2.7=230.5$. Now from Theorem 13.2.7, the first K-Banhatti index can be calculated as $\mathcal{B}_1(G_1 \circ G_2) = 1.1+1.2+1.1+0.5+0.8+1.2+0.8+0.6+0.8+0.5+0.8+0.7+1.1+0.6+1.1+0.7+1.1+1.2+1.1+0.7+4\times(0.1+0.5+0.1+0.4+0.4+0.1+0.4+0.5)+6\times(1.5\times 3+0.5\times 4)+6\times 0.7\times(5\times 3+2\times 4)+0.7\times 4\times 3\times(3\times 3-1)=230.5$.

Similarly after finding the corona product of fuzzy graphs G_1 and G_2 as given in Fig. 13.3 we can find $\mathcal{B}_2(G_1 \circ G_2) = 5.3\times3.3+5.3\times2.6+5.3\times3.3+5.3\times2.8+5\times3.3+5\times2.7+5\times2.6+5\times2.8+5.3\times2.8+5.3\times2.7+1.5\times1.1+1.5\times1.2+1.8\times1.2+1.8\times0.8+1.5\times1.1+1.5\times1.2+1.8\times1.2+1.8\times0.8+1.5\times1.1+1.5\times1.2+1.8\times1.2+1.8\times0.8+1.5\times1.1+1.5\times1.2+1.8\times1.2+1.8\times0.8+3\times1.1+3\times3.3+3.1\times1.2+3.1\times3.3+2.7\times0.8+2.7\times3.3+2.3\times1.1+2.3\times2.6+2.4\times1.2+2.4\times2.6+2\times0.8+2\times2.6+2.5\times1.1+2.5\times2.8+2.6\times1.2+2.6\times2.8+2.2\times0.8+2.2\times2.8+2.4\times1.1+2.4\times2.7+2.5\times1.2+2.5\times2.7+2.1\times0.8+2.1\times2.7 = 294.89$. Now from Theorem 13.2.8, the second K-Banhatti index can be calculated as $\mathcal{B}_2(G_1 \circ G_2) = (1.1\times1.2+1.1\times0.5+0.8\times1.2+0.8\times0.6+0.8\times0.5+0.8\times0.7+1.1\times0.6+1.1\times0.7+1.1\times1.2+1.1\times0.7)+4\times(0.1\times0.5+0.1\times0.4+0.4\times0.1+0.4\times0.5)+2\times0.7\times(4\times(0.5+1\times0.4+2\times0.5+1\times0.1)+3\times(4.9+2\times0.5\times4+2\times1.5\times3+3\times1.2+2\times0.5+3\times0.7+2\times0.6))+2\times(3\times(0.55+0.72)+4\times(0.17+0.04))+8\times1.5\times0.5+0.7^2\times(4\times5\times3^2+4\times4\times2+(3^2-1^2)\times4\times3) = 294.89$.

13.3 Algorithm for Banhatti indices

The algorithm to determine a fuzzy graph's first and second K-Banhatti indices is covered in this section.

Algorithm 13.3.1. Let $G = (\sigma, \mu)$ be a fuzzy graph with $\sigma^* = \{w_1, w_2, \cdots, w_n\}$.

1. Construct a matrix $A = [a_{ij}]$ with $a_{ij} = \mu(w_i w_j)$.
2. Construct a matrix $B = [b_{ij}]$ with $b_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} + \sum_{l=1}^{n} a_{jl} - 2a_{ij} & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$.
3. Construct a matrix $C = [c_{ij}]$ with $c_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} + b_{ij} & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$.
4. Construct a matrix $D = [d_{ij}]$ with $d_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} \times b_{ij} & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$.
5. Then $\mathcal{B}_1(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}$ and $\mathcal{B}_2(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$.

Illustration of algorithm 13.3.2. Let $G = (\sigma, \mu)$ be a fuzzy graph in Fig. 13.4 with $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\}$ and $\mu(w_1w_2) = 0.1$, $\mu(w_1w_3) = 0.4$, $\mu(w_2w_3) = 0.5$, $\mu(w_2w_6) = 0.4$, $\mu(w_3w_9) = 0.3$, $\mu(w_4w_5) = 0.6$, $\mu(w_4w_7) = 0.1$, $\mu(w_5w_8) = 0.3$, $\mu(w_6w_7) = 0.9$, $\mu(w_7w_8) = 0.2$, $\mu(w_8w_9) = 0.7$.

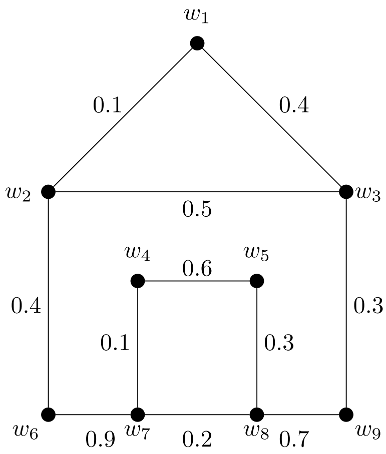

FIGURE 13.4

Fuzzy graph for the illustration for Algorithm.

Step 1: Construct a matrix $A = [a_{ij}]$ with $a_{ij} = \mu(w_i w_j)$. This is the matrix corresponding to the given fuzzy graph

$$A = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{array} \\ \begin{bmatrix} 0 & 0.1 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0.4 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.9 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0.2 & 0 & 0.7 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 \end{bmatrix} \end{array}$$

Step 2: Construct a matrix $B = [b_{ij}]$ with

$b_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} + \sum_{l=1}^{n} a_{jl} - 2a_{ij} & \text{if} \quad a_{ij} \neq 0 \\ 0 & \text{if} \quad a_{ij} = 0 \end{cases}$. This is a matrix showing corresponding fuzzy edge degrees.

$$B = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{array} \\ \begin{pmatrix} 0 & 1.3 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.3 & 0 & 1.2 & 0 & 0 & 1.5 & 0 & 0 & 0 \\ 0.9 & 1.2 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 1.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 1.7 & 0 & 0.7 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1.5 & 0 & 2 & 0 & 0.8 \\ 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0.8 & 0 \end{pmatrix} \end{array}$$

Step 3: Construct a matrix $C = [c_{ij}]$ with $c_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} + b_{ij} & \text{if} \quad a_{ij} \neq 0 \\ 0 & \text{if} \quad a_{ij} = 0 \end{cases}$.

$$C = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{array} \\ \begin{pmatrix} 0 & 1.8 & 1.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.3 & 0 & 2.2 & 0 & 0 & 2.5 & 0 & 0 & 0 \\ 2.1 & 2.4 & 0 & 0 & 0 & 0 & 0 & 0 & 2.8 \\ 0 & 0 & 0 & 0 & 1.1 & 0 & 2.4 & 0 & 0 \\ 0 & 0 & 0 & 1.3 & 0 & 0 & 0 & 2.4 & 0 \\ 0 & 2.8 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2.9 & 0 & 1.9 & 0 & 3.2 & 0 \\ 0 & 0 & 0 & 0 & 2.7 & 0 & 3.2 & 0 & 2 \\ 0 & 0 & 2.6 & 0 & 0 & 0 & 0 & 1.8 & 0 \end{pmatrix} \end{array}$$

Step 4: Construct a matrix $D = [d_{ij}]$ with $d_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} \times b_{ij} & \text{if} \quad a_{ij} \neq 0 \\ 0 & \text{if} \quad a_{ij} = 0 \end{cases}$

$$D = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{array} \\ \begin{pmatrix} 0 & 0.65 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.3 & 0 & 1.2 & 0 & 0 & 1.5 & 0 & 0 & 0 \\ 1.08 & 1.44 & 0 & 0 & 0 & 0 & 0 & 0 & 1.92 \\ 0 & 0 & 0 & 0 & 0.28 & 0 & 1.19 & 0 & 0 \\ 0 & 0 & 0 & 0.36 & 0 & 0 & 0 & 1.35 & 0 \\ 0 & 1.95 & 0 & 0 & 0 & 0 & 0.91 & 0 & 0 \\ 0 & 0 & 0 & 2.04 & 0 & 0.84 & 0 & 2.4 & 0 \\ 0 & 0 & 0 & 0 & 1.8 & 0 & 2.4 & 0 & 0.96 \\ 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0.8 & 0 \end{pmatrix} \end{array}$$

Step 5: In this step the first and second K-Banhatti indices are found. Now the first and second K-Banhatti indices are the sum of all entries in C and D, respectively, i.e., $\mathcal{B}_1(G) = 1.8 + 1.4 + 2.3 + 2.2 + 2.5 + 2.1 + 2.4 + 2.8 + 1.1 + 2.4 + 1.3 + 2.4 + 2.8 + 2 + 2.9 + 1.9 + 3.2 + 2.7 + 3.2 + 2 + 2.6 + 1.8 = 49.8$ and $\mathcal{B}_2(G) = 0.65 + 0.45 + 1.3 + 1.2 + 1.5 + 1.08 + 1.44 + 1.92 + 0.28 + 1.19 + 0.36 + 1.35 + 1.95 + 0.91 + 2.04 + 0.84 + 2.4 + 1.8 + 2.4 + 0.96 + 1.6 + 0.8 = 28.42$.

13.4 Application

Let A be a organization, with W and M representing two distinct teams within the organization. It is assumed that both team W and team M are provided with identical data sets for analysis. In the context of this analysis, two significant factors come into play: "pre-knowledge" and "knowledge sharing." The term "pre-knowledge" refers to the understanding or expertise that each team member possesses before engaging in the analysis. This pre-existing knowledge is not uniform across all members; it varies based on their backgrounds, experiences, and capabilities. This diversity in pre-knowledge is captured by the concept of fuzzy subset, symbolized by σ. On the other hand, "knowledge sharing" refers to the exchange of insights and information among team members during discussions. This interplay of ideas contributes to the overall analytical process. The dynamics of this knowledge sharing are described by a fuzzy relation, designated as μ. This fuzzy relation outlines how effectively information is communicated between any two individuals during their discussions. Now, let's consider the concept of vertex. In a graph, a vertex is a point that represents an entity, often linked to other vertices by edges. In this context, a vertex represents a team member, and the edges between vertices symbolize the knowledge shared during interactions. In this context, the degree of an arbitrary vertex equates to the cumulative knowledge shared by a team member through their interactions with fellow members. During discussions involving a pair of team members, a trainee is involved to ensure the quality of the shared knowledge. This trainee reviews and cross-references the information exchanged by these two individuals with the rest of the team, excluding the immediate pair. This cross-referencing process corresponds to degree of the edge connecting these two team members. In essence, it quantifies the depth and accuracy of knowledge shared between them. By considering all potential combinations of team members and trainees, the cumulative output of knowledge sharing is computed by summing the contributions from both the individual team members and the assisting trainees in each such case. The aggregation of these knowledge-sharing outputs parallels the essence of the first K-Banhatti index within the plotted fuzzy graph. Given that effective discussions among team members inherently involve the sharing of at most one person's existing knowledge, this scenario aptly aligns with the concept of a complete fuzzy graph. Leveraging Theorem 13.1.14, the computation of the first K-Banhatti index within the graphical representation becomes an expedited endeavor. The theorem provides a formula to efficiently calculate this index, which characterizes the intensity of knowledge sharing within the fuzzy graph. By constructing separate fuzzy graphs for team W and team M, a comparison can be drawn between their respective outputs. The team that contributes more to the analysis is determined by evaluating their first K-Banhatti index. In simpler terms, a higher first K-Banhatti index signifies a team with more substantial output.

13.5 Exercises

1. Calculate first and second Banhatti indices of the fuzzy graph $G = (\sigma, \mu)$ given by $\sigma^* = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$, $\sigma(x) = 1$ for every $x \in \sigma^*$ and $\mu(w_1w_2) = 0.3$, $\mu(w_1w_3) = 0.1$, $\mu(w_2w_3) = 0.4$, $\mu(w_2w_4) = 0.3$, $\mu(w_2w_5) = 0.5$, $\mu(w_3w_6) = 0.6$, $\mu(w_3w_7) = 0.2$, $\mu(w_5w_6) = 0.1$.

2. Let G be a fuzzy tree and F its unique spanning tree. Show that the first and second Banhatti indices of F are always bounded by that of G.

3. In Example 13.2.6, replace all weights by one. Calculate the Banhatti indices of the coalescence of G_1 and G_2. What do you observe?

CHAPTER

14 Generalized cycle connectivity of fuzzy graphs ✪

The cycle connectivity of a network allows us to determine the cyclic flow and hence to determine the cyclic reachability of the network. Cycle connectivity depends on the strength of all strong cycles in a fuzzy graph. As a result, certain classes of fuzzy graphs, like fuzzy trees, that lack strong cycles are left behind. Due to the presence of cycles in fuzzy trees, it is important to consider the cyclic flow in a fuzzy tree network. The purpose of this chapter is to study a generalized version of cycle connectivity in fuzzy graphs, which takes into account of the strengths of all cycles in a fuzzy graph. This chapter entirely depends on [7]. Basic works on cycle connectivity can be seen in [16,50,64,83].

14.1 Generalized cycle connectivity

In this section, generalized cycle connectivity in fuzzy graphs are studied, by considering all cycles in the fuzzy graph.

Consider a fuzzy graph $G = (\sigma, \mu)$. The set

$$\theta^g(u, v) = \{\alpha : \alpha \text{ is the strength of a cycle containing both } u \text{ and } v\}$$

is called the θ^g**- evaluation** of u and v.

If there is no cycle containing a given pair of vertices u and v of G, then $\theta^g(u, v) = \phi$.

Definition 14.1.1. **Generalized cycle connectivity** (GCC) between u and v in a fuzzy graph G is given by

$$C_D^g(u, v) = \text{Max } \{\alpha : \alpha \in \theta^g(u, v),\ u, v \in \sigma^*\}.$$

If $\theta^g(u, v) = \emptyset$ for some u and v in σ^*, then it indicates that there is no cycle containing u and v in D, and hence we set $C_D^g(u, v) = 0$.

Example 14.1.2. For a fuzzy graph $D = (\sigma, \mu)$ with vertex set $\sigma^* = \{u_1, u_2, u_3, u_4, u_5\}$, $\mu(u_1u_2) = 0.5$, $\mu(u_1u_3) = 0.6$, $\mu(u_2u_3) = 0.3$, $\mu(u_2u_4) = 0.8$, $\mu(u_3u_4) = 0.7$, $\mu(u_4u_5) = 1$, $\mu(u_2u_5) = 0.3$ and $\mu(u_3u_5) = 0.2$ (Fig. 14.1).

✪ This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00022-2

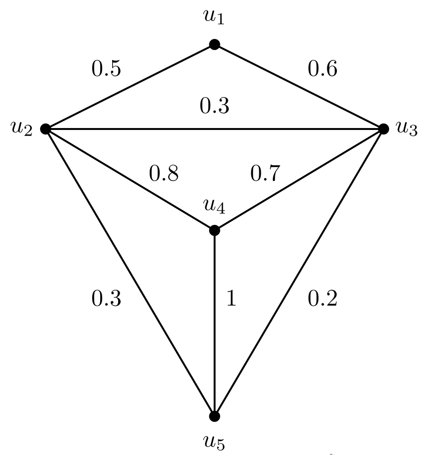

FIGURE 14.1

Fuzzy graph G with $C_G^g(u_1, u_4) = 0.5$.

There are three cycles containing vertices u_1 and u_4 namely $u_1u_2u_4u_3u_1$, $u_1u_2u_4u_5u_3u_1$, and $u_1u_2u_5u_4u_3u_1$. Hence $\theta^g(u_1, u_4) = \{0.5,0.2,0.3\}$ and $C_G^g(u_1, u_4) = 0.5$.

GCC of a fuzzy graph G is defined as

$$C^g(G) = \text{Max } \{C_G^g(u, v) : u, v \in \sigma^*\}$$

That is, GCC of G is the strength of its strongest cycle.

It is obvious that $C^g(G)$ is never greater than the strength of connectedness between any two vertices of G.

Since in a graph, each cycle is of strength 1, $C^g(G) = 1$ if G is cyclic and $C^g(G) = 0$ if G is a tree.

Clearly, given a fuzzy graph $G = (\sigma, \mu)$, we have $CC(G) \leq C^g(G)$.

Proposition 14.1.3. *For a partial fuzzy subgraph $H = (\tau, \nu)$ of a fuzzy graph $G = (\sigma, \mu)$, $C^g(H) \leq C^g(G)$.*

Theorem 14.1.4. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph with vertex set $\sigma^* = \{u_1, u_2, ..., u_n\}$ such that $\sigma(u_i) = m_i$, $(i = 1, 2, ..., n)$ where m_i's are in the increasing order of vertex strengths. Then, $C^g(G) = m_{n-2}$.*

Proof. Given G is complete. Then any 3 vertices of G are in a 3-cycle. Therefore in order to find cycles of maximum strength in G, it is enough to find all the 3-cycles of maximum strength in G. For, let $C = u_1u_2u_3u_4$ be a 4-cycle of strength p in G. Consider the 3-cycles $C_1 = u_2u_4u_1u_2$ and $C_2 = u_2u_4u_3u_2$ in C. Such 3-cycles exist as G is a CFG. Since C is a cycle of strength p, $\mu(u_iu_j) \geq p$, $\forall$ edge u_iu_j in C. Specifically, $\mu(u_1u_4) \geq p$ and $\mu(u_1u_2) \geq p$. Hence, $\mu(u_2u_4) \geq \text{Min}\{\mu(u_1u_4),$

$\mu(u_1u_2)\}$ as there are no δ-edges in G. Thus $\mu(u_2u_4) \geq p$. Suppose $\mu(u_2u_4) = p$. Then C_1 and C_2 have strength p.

Now if $\mu(u_2u_4) > p$, then at least one of C_1 or C_2 will have strength p.

Thus, in both cases, strength of C is the minimum of the strengths of C_1 and C_2. That is, strengths of a 4-cycle and a 3-cycle are same in G.

Clearly, the 3-cycle formed by the vertices of maximum weight will have the maximum strength in G. Thus $C = u_{n-2}u_{n-1}u_nu_{n-2}$ is a cycle of highest strength with strength s$(C) = m_{n-2} \wedge m_{n-1} \wedge m_n = m_{n-2}$. Hence, $C^g(G) = m_{n-2}$. ■

It follows from the above mentioned theorem that a complete fuzzy graph G exists, with $C^g(G) = m$, for every $m \in (0, 1]$.

Suppose G is a fuzzy tree, then $C_{u,v}^G = 0$ for every couple of vertices u and v in G and hence $CC(G) = 0$. Since a fuzzy tree may contain cycles, we get the result given below.

Theorem 14.1.5. *Given a fuzzy tree $G = (\sigma, \mu)$. Then*

$$C^g(G) = \begin{cases} 0, & G \text{ is a tree} \\ w_\delta, & \text{otherwise} \end{cases}$$

where w_δ is the maximum weight of δ-edges in G.

Proof. Suppose G is a tree. Then, it is acyclic and hence $C^g(G) = 0$. Now suppose G is cyclic. As a fuzzy tree, G lacks strong cycles. Thus, every cycle of G has at least one δ-edge, and every such edge of G exists in at least one cycle in G. Let w_δ be the highest weight of δ-edges in G. Then, there is a cycle C in G of strength w_δ. Since the strength of each cycle in G is same as the weight of the weakest δ-edge in it, we have $C^g(G) = w_\delta$. ■

It is clear that a fuzzy cycle has no δ-edges. Hence GCC and cycle connectivity of a fuzzy cycle G are equal to the strength of G.

Theorem 14.1.6 proves that the g-cycle connectivity of isomorphic fuzzy graphs are equal.

Theorem 14.1.6. *Consider two fuzzy graphs $M = (\sigma, \mu)$ and $K = (\tau, \nu)$. If M and K are isomorphic, then $C^g(M) = C^g(K)$.*

Proof. Since M and K are isomorphic, there is a bijection, say $h : \sigma^* \to \tau^*$. For $\sigma^* = \{u_1, u_2, \cdots, u_n\}$, let $h(u_j) = v_j$ for $j = 1, 2, \cdots, n$. In that case,

$$\mu(u_iu_j) = \nu(h(u_i)h(u_j)) = \nu(v_iv_j).$$

Suppose that $\mathcal{C}_M = \{u_1, u_2, \cdots, u_p\}$, $p \leq n$, is a cycle of largest strength in M. That is, $s(\mathcal{C}_M) = C^g(M)$. Then, $\mathcal{C}_N = \{v_1, v_2, \cdots, v_p\}$, $p \leq n$, is a cycle in K. It is left to show that $\mathcal{C}_K$ has the highest strength in K. Suppose there exists a cycle $\tilde{\mathcal{C}}_K$ in K with $s(\tilde{\mathcal{C}}_K) > s(\mathcal{C}_K)$. Let $\tilde{\mathcal{C}}_K = \{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_q\}$, $q \leq n$. Then, corresponding to each $\tilde{v}_j \in \tau^*$, there exists $u_j \in \sigma^*$ such that $h(u_j) = \tilde{v}_j$, $j = 1, 2, \cdots, q$, $q \leq n$. Since isomorphism on fuzzy graphs preserves vertex strength and weight of the edges, the cycle $\{u_1, u_2, ..., u_q\}$ is supposed to be a cycle of highest strength in M. But this results in a contradiction. Hence, $\mathfrak{C}_K$ is a cycle of highest strength in K with $C^g(K) = s(\mathfrak{C}_K) = s(\mathfrak{C}_M) = C^g(M)$. ■

Theorem 14.1.7. *Consider a fuzzy graph G. Suppose that K is a fuzzy graph formed by removing an edge from G. If $C^g(K) < C^g(G)$, then G contains a unique cycle $\mathfrak{C}$ of highest strength.*

Proof. Assume that G contains at least two cycles, C and if possible, $\hat{C}$ of maximum strength. Then either C or $\hat{C}$ are disjoint or they have at least one vertex in common. In either cases, deletion of an edge e from C does not affect $s(\hat{C})$. Let $K = G - \{e\}$. Then, $C^g(K) = s(\hat{C}) = C^g(G)$, which is a contradiction. Hence, G has a unique cycle of maximum strength. ■

A fuzzy graph $H = (\sigma', \mu')$ is considered a super fuzzy graph of a fuzzy graph G if G is a fuzzy subgraph of H.

Theorem 14.1.8. *Consider a fuzzy graph $G = (\sigma, \mu)$ with g-cycle connectivity $C^g(G)$. Then, a super fuzzy graph J of G can be constructed with $C^g(J) \geq C^g(G)$.*

Proof. If all edges of a fuzzy graph G are of equal weight, then clearly G is a super graph of itself and $C^g(G) \geq C^g(G)$.

Suppose $G = (\sigma, \mu)$ is a fuzzy graph with $\sigma^* = \{u\}$. Then $C^g(G) = 0$. Construct a fuzzy graph $J = (\tilde{\sigma}, \tilde{\mu})$ with $\tilde{\sigma}^* = \sigma^* \cup \{v, w\}$ and $\tilde{\mu}^* = \{uv, vw, uw\}$ given as

$$\tilde{\sigma}(y) = \begin{cases} \sigma(y) & : \text{if } y = u \\ 1 & : \text{if } y = v, w \end{cases}$$

and

$$\tilde{\mu}^*(xy) = p, \text{ for some } p \in (0, 1],\ xy \in \tilde{\mu}^*.$$

That is, J is the cycle $uvwu$. Thus in this case, it is evident that $C^g(J) = s(J) > 0 = C^g(G)$.

Now, suppose that $\sigma^* = \{u, v\}$ with $\mu^* = \{uv\}$. Here, J can be constructed by adding a vertex w such that $\tilde{\sigma}^* = \{u, v, w\}$ and $\tilde{\mu}^* = \{uv, vw, uw\}$. Rest of the proof follows from the above case when $\sigma^* = \{u\}$.

Finally, suppose that $\sigma^* = \{u_1, u_2, \cdots, u_n\}$. Let $u_k u_{k+1}$, for $k < n$, be an edge of maximum weight in G with $\mu(u_k u_{k+1}) = p$. Then $C^g(G) \leq p$. Consider the sets $\tilde{\sigma}^* = \sigma^* \cup \{v\}$ and $\tilde{\mu}^* = \mu^* \cup \{vu_1, vu_2, \cdots, vu_n\}$. Then $J = (\tilde{\sigma}, \tilde{\mu})$ is a super fuzzy graph of G. Assign $\tilde{\mu}^*(vu_i) = \vee\{\mu(u_i u_j) : u_j \in \sigma^*,\ 1 \leq i \leq n\}$. Then the cycle $vu_k u_{k+1} v$ is of strength p, which is the largest. Hence, $C^g(J) = p \geq C^g(G)$. ■

14.2 Generalized cyclic cutvertices and bridges

This section focuses on the edges and vertices of a fuzzy graph G whose removal change the g-cycle connectivity of G. A few results regarding the existence of such vertices are also given.

Definition 14.2.1. Consider a fuzzy graph $G = (\sigma, \mu)$. A vertex $u \in \sigma^*$ is called a ***g*-cyclic cutvertex** of G if

$$C^g(G - u) < C^g(G).$$

In other words, if the removal of a vertex u from a fuzzy graph G reduces the GCC of G, then u is a g-cyclic cutvertex.

Similarly, a ***g*-cyclic bridge** of G is an edge $uv \in \mu^*$ such that

$$C^g(G - uv) < C^g(G).$$

That is, a g-cyclic bridge of a fuzzy graph G is that edge whose deletion reduces the GCC of G.

Example 14.2.2. Consider an octahedral fuzzy graph G $= (\sigma, \mu)$ with $\sigma^* = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mu(u_1u_2) = 0.6$, $\mu(u_1u_3) = 0.9$, $\mu(u_1u_5) = \mu(u_4u_5) = 0.3$, $\mu(u_1u_6) = \mu(u_2u_3) = 0.7$, $\mu(u_2u_4) = 0.2$, $\mu(u_2u_6) = 0.8$, $\mu(u_3u_4) = \mu(u_3u_5) = 0.5$, $\mu(u_4u_6) = 0.2$, and $\mu(u_5u_6) = 0.2$ (Fig. 14.2).

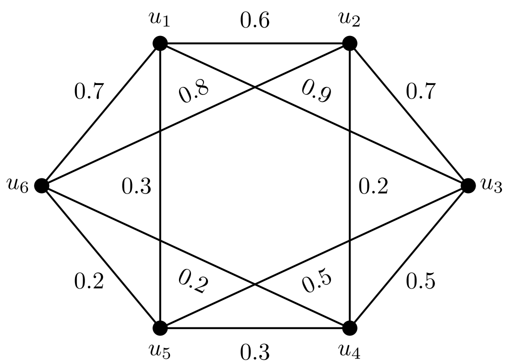

FIGURE 14.2

Fuzzy graph containing g-cyclic cutvertices and bridges.

Here, $C^g(G) = 0.6$. Thus vertices u_1 and u_2 are g-cyclic cutvertices and u_1u_2 is the g-cyclic bridge of G.

Note that a g-cyclic cutvertex need not be a cyclic cutvertex and a g-cyclic bridge need not be a cyclic bridge.

Proposition 14.2.3. *Suppose that uv is a g-cyclic bridge of a fuzzy graph G. Then both u and v are g-cyclic cutvertices of G.*

Proof. Let edge uv be a g-cyclic bridge of G. Then, $C^g(G-uv) < C^g(G)$.

Since $C^g(G-u) \leq C^g(G-uv)$, we have $C^g(G-u) < C^g(G)$. Similarly, $C^g(G-v) < C^g(G)$ and hence u and v are g-cyclic cutvertices of G. ■

It is noticeable that all vertices and edges of a fuzzy graph are, respectively, its g-cyclic cutvertices and g-cyclic bridges if the underlying graph is a cycle. The following result is based on this idea and its proof is omitted.

Proposition 14.2.4. *Suppose G is a fuzzy graph such that G^* is a cycle. Then, every vertex and edge of G are, respectively, g-cyclic cutvertices and g-cyclic bridges of G.*

Theorem 14.2.5 gives a necessary and sufficient condition for the existence of g-cyclic cutvertices in a fuzzy graph.

Theorem 14.2.5. *A vertex $r \in \sigma^*$ in a fuzzy graph $G = (\sigma, \mu)$ is a g-cyclic cutvertex if and only if r lies on all cycles with maximum strength.*

Proof. Suppose that u is a g-cyclic cutvertex of G. Then $C^g(G-u) < C^g(G)$. That is, deletion of u removes all cycles in G with maximum strength. Hence, u is a vertex common to all cycles with largest strength in G.

Conversely, suppose that u lies on all cycles with greatest strength. Then $G-u$ removes all such cycles and hence $C^g(G-u) < C^g(G)$. That is, u is a g-cyclic cutvertex of G. ■

Since all cycles are strong in a CFG, the condition for the existence of g-cyclic cutvertices (or g-cyclic bridges) is the same as that of cyclic cutvertices (or cyclic bridges), and hence we have the following result. Note that the existence of g-cyclic cutvertex in a CFG consisting of three vertices is trivial.

Theorem 14.2.6. *Consider a complete fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{u_1, u_2, ..., u_n\}$, $n \geq 4$. Let $\sigma(u_i) = m_i$, $1 \leq i \leq n$ where m_i's are in the increasing order of vertex strength. Then G contains a g-cyclic cutvertex (or a g-cyclic bridge) if and only if $m_{n-3} < m_{n-2}$.*

Proof. Suppose that v is a g-cyclic cutvertex of G. Then $C^g(G-s) < C^g(G)$ which implies that v lies in a unique cycle C of maximum strength. Since $m_1 \leq m_2 \leq ... \leq m_n$, the cycle $C = u_{n-2}u_{n-1}u_n$ is of maximum strength m_{n-2}, and $v \in \{u_{n-2}, u_{n-1}, u_n\}$.

It is to prove that $m_{n-3} < m_{n-2}$. If possible, suppose the contrary. That is $m_{n-3} = m_{n-2}$. Then the cycles $C_1 = u_{n-2}u_{n-1}u_n$ and $C_2 = u_n u_{n-1} u_{n-3}$ will have the same strength m_{n-2} and the removal of vertices u_{n-2}, u_{n-1}, or u_n will not reduce $C^g(G)$. This goes against our presumption that v is a g-cyclic cutvertex of G. Thus $m_{n-3} < m_{n-2}$.

Conversely, suppose that $m_{n-3} < m_{n-2}$. Since $m_{n-2} \leq m_{n-1} \leq m_n$ and $m_{n-3} < m_{n-2}$, the cycle $C = u_{n-2}u_{n-1}u_n$ will have the maximum strength. Hence removal of u_n, u_{n-1}, or u_{n-2} will reduce $C^g(G)$ and thus the vertices of C are the g-cyclic cutvertices of G. ■

The above theorem makes it obvious that a CFG contains three g-cyclic cutvertices (or g-cyclic bridges) if it exists.

For a complete fuzzy graph, the concept of cycle connectivity and g-cycle connectivity coincides. Hence, the set of g-cyclic cutvertices (or g-cyclic bridges) is the same as the set of cyclic cutvertices (or cyclic bridges). Similar is the case with fuzzy cycles. Theorem 14.2.7 shows the existence of fuzzy graphs other than CFGs and fuzzy cycles whose cyclic cutvertices and g-cyclic cutvertices coincide.

Theorem 14.2.7. *Consider a fuzzy graph G containing exactly one cycle S of maximum strength. Let P denote the set of cyclic cutvertices and Q denote the set of g-cyclic cutvertices of G. Then $P = Q$ if and only if S is a strong cycle.*

Proof. Assume that $P = Q$. Let $z \in P$. Then $z \in Q$. That is, z is a g-cyclic cutvertex of G. Hence z lies in S and $C^g(G - z) < C^g(G)$. Also z is a cyclic cutvertex and hence it lies in some strong cycle S' of maximum strength.

Suppose $S \neq S'$. Then there is at least one vertex y that is a member of either S or S' but not both. This contradicts our assumption that $P = Q$. Hence, S is a strong cycle.

For the converse, assume that S is a strong cycle. Being the only cycle of maximum strength, vertices of S are the g-cyclic cutvertices of G. Also since S is strong, vertices of S are the cyclic cutvertices of G since the absence of any vertex of S decreases the cycle connectivity of G. ■

Definition 14.2.8. A fuzzy graph G is termed **cyclically stable** if it lacks both g-cyclic cutvertices and g-cyclic bridges.

14.3 g-Cyclic vertex connectivity and edge connectivity

This section discusses the characteristics of the set of vertices and edges whose removal minimizes the GCC of a fuzzy graph. A set of vertices $W \subseteq \sigma^*$ is known as **g-cyclic vertex cut** of the fuzzy graph $G = (\sigma, \mu)$ if $C^g(G - W) < C^g(G)$, provided G is not a tree.

Similar to a g-cyclic vertex cut, we can refer a collection of edges $E \subseteq \mu^*$ as a **g-cyclic edge cut** of G if $C^g(G - E) < C^g(G)$.

Example 14.3.1. Consider a fuzzy graph $G = (\sigma, \mu)$ (Fig. 14.3) with $\sigma^* = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\mu(u_1u_2) = 0.32$, $\mu(u_2u_6) = 0.75$, $\mu(u_6u_3) = \mu(u_1u_4) = 0.55$, $\mu(u_3u_4) = 0.75$, $\mu(u_1u_5) = \mu(u_4u_5) = 0.65$, and $\mu(u_5u_6) = 0.85$. There are two cycles $u_1u_5u_4u_1$ and $u_5u_6u_3u_4u_5$ of strength 0.55 which is equal to $C^g(G)$. Here the set $W = \{u_3, u_4\}$ is a g-cyclic vertex cut and the set $E = \{u_1u_4, u_3u_6\}$ is a g-cyclic edge cut of G.

Consider a g-cyclic vertex cut W of a fuzzy graph G. The **strong weight** S^g of W is given as

$$S^g(W) = \sum_{u \in W} \mu(uv),$$

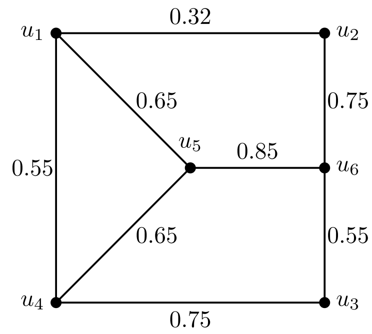

FIGURE 14.3

Fuzzy graph with a g-cyclic vertex cut and a g-cyclic edge cut.

where $\mu(uv)$ is the weight of the weakest strong edge incident on u.

Similarly, a strong weight $\overline{S^g}$ of a g-cyclic edge cut E of G is defined as

$$\overline{S^g}(E) = \sum_{e_j \in E} \mu(e_j),$$

where e_j is a strong edge in E.

Definition 14.3.2. Consider a fuzzy graph G. g**-cyclic vertex connectivity** of G denoted by $\kappa^g(G)$ is defined as the minimum of strong weights of all g-cyclic vertex cuts of G.

Definition 14.3.3. For a fuzzy graph G, g**-cyclic edge connectivity** $\overline{\kappa^g}(G)$ is defined as the minimum of the nonzero strong weights of all g-cyclic edge cuts of G.

In Example 14.3.1, $W_1 = \{u_3, u_4\}$ and $W_2 = \{u_1, u_4\}$ are 2-gCVCs of G with $S^g(W_1) = 0.65 + 0.75 = 1.4$, $S^g(W_2) = 0.65 + 0.65 = 1.3$. Also, $\{u_5\}$ is a g-cyclic vertex cut with $S^g(\{u_5\}) = 0.65$, and hence $\kappa^g(G) = 0.65$.

The following result is evident and hence the proof is omitted.

Theorem 14.3.4. *Consider a fuzzy graph $G = (\sigma, \mu)$. Let $H = (\tau, \nu)$ be a partial fuzzy subgraph of G. Then $\kappa^g(H) \leq \kappa^g(G)$.*

The following theorem gives the g-cyclic vertex connectivity of a complete fuzzy graph.

Theorem 14.3.5. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph with $\sigma^* = \{u_1, u_2, ..., u_n\}$ such that $\sigma(u_i) = m_i$, where m_i's are in the increasing order of vertex strength. Then $\kappa^g(G) = m_1$.*

Proof. Consider a fuzzy graph G with vertex set $\sigma^* = \{u_1, u_2, ..., u_n\}$ such that $\sigma(u_i) = m_i$, where $m_1 \leq m_2 \leq \cdots \leq m_n$, m_1 being the lowest. Let S be the cycle of maximum strength in G. Then $C^g(G)$ = strength of S. Hence, each vertex of S is a g-cyclic cutvertex of G.

Since there exists at least one edge of weight m_1 incident to each vertex u_i ($1 \leq i \leq n$), we have $S^g(\{x\}) = m_1$, where x is a g-cyclic cutvertex of G. Thus $\kappa^g(G) = m_1$, since m_1 is the minimum of strong weights of all g-cyclic cutvertices of G. ■

Next theorem gives a relationship between the g-cyclic vertex connectivity and vertex connectivity of a complete fuzzy graph.

Theorem 14.3.6. *Let G represent a complete fuzzy graph. Then* $\kappa^g(G) \leq \kappa(G)$.

Proof. Consider a complete fuzzy graph $G = (\sigma, \mu)$ where $\sigma^* = \{u_1, u_2, ..., u_n\}$ with $d_s(u_1) \leq d_s(u_2) \leq ... \leq d_s(u_n)$. Then $d_s(u_1) = \delta_s(G)$. We consider two cases:

Case 1. u_1 is a g-cyclic cutvertex (Fig. 14.4).

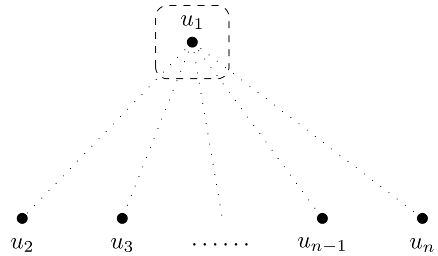

FIGURE 14.4

Illustration to Case 1 of Theorem 14.3.6.

Then $X = \{u_1\}$ is a g-cyclic vertex cut of G. We have,

$$\begin{aligned} S^g(X) &= \wedge\{\mu(u_1u_j)\}, 2 \leq j \leq n, \text{ where } \wedge \text{ denotes minimum} \\ &\leq \sum \mu(u_1u_j) \\ &= \delta_s(G) \end{aligned}$$

Since $\kappa^g(G) = \min\{S^g(X)\}$, where X is a g-cyclic vertex cut of G, we have,

$$\kappa^g(G) \leq S^g(X) \leq \delta_s(G) = \kappa(G).$$

Case 2. u_1 is not a g-cyclic cutvertex (Fig. 14.5).

Choose a g-cyclic vertex cut $Y = \{y_1, y_2, ..., y_m\}$ of G such that $S^g(Y) = \kappa^g(G)$. It follows that

$$\begin{aligned} \kappa^g(G) &= S^g(Y) \\ &= \sum_{i=1}^{m} \min\{\mu(y_iu_j)\} \ \forall u_j \in \sigma^*, \ i \neq j, \ j = 1, 2, ..., n. \end{aligned}$$

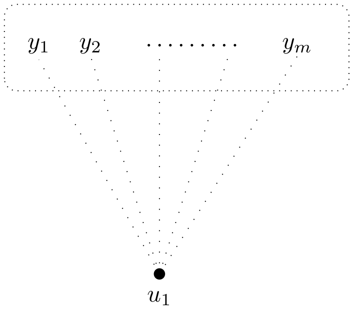

FIGURE 14.5

Illustration to Case 2 (Theorem 14.3.6).

$$= \sum_{i=1}^{m} \mu(y_i u_1)$$
$$\leq d_s(u_1)$$
$$= \delta_s(G)$$
$$= \kappa(G)$$

Hence, in both cases, $\kappa^g(G) \leq \kappa(G)$. ∎

The upcoming result gives a version of Whitney's theorem using g-cycle connectivity in fuzzy graphs.

Theorem 14.3.7. *Let G be a complete fuzzy graph. Then*

$$\kappa^g(G) \leq \overline{\kappa^g}(G) \leq \delta_s(G).$$

Proof. Let $C_1, C_2, ..., C_n$ be cycles of maximum strength in G. That is, these are the cycles with strength equal to the g-cycle connectivity of G. Consider the set $E = \{e_1, e_2, ..., e_n\}$ where edge $e_i = u_i v_i$ belongs to cycle C_i, $1 \leq i \leq n$. Then certainly E is a g-cyclic edge cut of G. Let $\overline{S^g}(E)$ be the strong weight of E. Then it follows from the definition of g-cyclic edge connectivity that

$$\overline{\kappa^g}(G) \leq \overline{S^g}(G).$$

Consider the set $Y = \{y_1, y_2, ..., y_n\}$ where y_i is an end vertex of the edge $e_i = x_i y_i$, $1 \leq i \leq n$. Then Y is a g-cyclic vertex cut of G. Let $S^g(Y)$ be the strong weight of Y. Then,

$$S^g(Y) \leq \overline{\kappa^g}(G).$$

This is clear from the definition of g-cyclic vertex connectivity of G. Since $\kappa^g(G) \leq S^g(Y)$, we have,

$$\kappa^g(G) \leq S^g(Y)$$

$$\leq \overline{\kappa^g}(G)$$
$$\leq \delta_s(G).$$

Thus $\kappa^g(G) \leq \overline{\kappa^g}(G) \leq \delta_s(G)$. ■

For a fuzzy tree G, $\kappa(G)$ is literally the smallest weight of all strong edges in G [63]. Similarly, we can relate $\kappa^g(G)$ to the weights of strong edges in G as in the theorem below.

Theorem 14.3.8. *Consider a fuzzy tree G which is cyclic. Let C be the only cycle of maximum strength in G. Then $\kappa^g(G) = \overline{\kappa^g}(G) = \wedge\{\mu(uv) : uv$ is a strong edge in $C\}$.*

Proof. Let G be a fuzzy tree with a unique maximum spanning tree F. Then all edges of F are strong and for an edge uv not in F, there exists a path P joining u and v in F whose strength is greater than $\mu(uv)$. Such an edge exists since G is cyclic.

Let uv be an edge of G that does not belong to F such that the cycle C formed by uv and the $u - v$ path in F is of greatest strength. Then all edges of C are g-cyclic bridges and hence are the g-cyclic edge cuts of G. Thus, $\overline{\kappa^g}(G)$ is the minimum weight of all strong edges in C. Since all vertices of C are g-cyclic cutvertices, we have $\kappa^g(G) = \overline{\kappa^g}(G) = \wedge\{\mu(uv) :$ vv is a strong edge in $C\}$. ■

Note that for a fuzzy tree G that is not cyclic, $\kappa^g(G) = 0$.

Since a fuzzy tree G lacks strong cycles, $\kappa_c(G) = 0$ and hence $\kappa_c(G) \leq \kappa(G)$. But the case is different for g-cyclic vertex connectivity. Clearly, $\kappa^g(G) = 0$ and hence $\kappa^g(G) \leq \kappa(G)$ for a fuzzy tree that is acyclic. The following theorem shows the case when G is cyclic.

Theorem 14.3.9. *Let G be a cyclic fuzzy tree. Then $\kappa^g(G) \geq \kappa(G)$.*

Proof. Consider a cyclic fuzzy tree G. Let F be the unique maximum spanning tree of G. Since G is cyclic, there exist at least one edge not in F. Clearly F contains only strong edges and let uv be a strong edge of minimum weight $\tilde{w}$. Then $\kappa(G) = \mu(uv) = \tilde{w}$. Let C be a cycle of largest strength. There are two cases to consider:

Case 1. C is unique.

Suppose that C is the only cycle of maximum strength. There are two possibilities that occur in this case:

Case 1.1. C contains the edge uv.

Since C is the only cycle of maximum strength, all vertices of C are g-cyclic cutvertices and hence by Theorem 14.3.8, $\kappa^g(G) = \mu(uv) = \tilde{w}$.

Thus $\kappa^g(G) = \kappa(G) = \tilde{w}$.

Case 1.2. C does not contain the edge uv.

Suppose that uv lies outside the cycle C. Then all strong edges in C have weight greater than $\tilde{w}$. Hence, $\kappa^g(G) > \tilde{w}$.

Thus in either cases, $\kappa^g(G) \geq \kappa(G)$.

Case 2. ***C* is not unique.**

Suppose there exists more than one cycle of maximum strength. Let C' be a cycle of maximum strength. If C and C' have a vertex in common, then it is a g-cyclic cutvertex and hence $\kappa^g(G) \geq \mu(uv) = \tilde{w}$, since uv is a strong edge of minimum weight.

If C and C' does not have any vertex in common, then clearly every g-cyclic vertex cut of G contains at least two vertices. Hence, the strong weight of all g-cyclic vertex cuts exceeds $\tilde{w}$ and thus $\kappa^g(G) \geq \tilde{w}$. Thus both the cases conclude that $\kappa^g(G) \geq \kappa(G)$. ■

We can define a ***g*-cyclic end vertex** as the vertex u that lies on a cycle which is not a g-cyclic cutvertex of G. It is known that a cyclic cutvertex is never a fuzzy end vertex [13]. But the situation differs in the case of g-cyclic cutvertices. A g-cyclic cutvertex can be a fuzzy end vertex as seen in the example below.

Example 14.3.10. Let $G = (\sigma, \mu)$ be a fuzzy graph with $\sigma^* = \{u_1, u_2, u_3, u_4, u_5\}$ such that $\mu(u_1u_2) = \mu(u_2u_5) = 0.5$, $\mu(u_1u_3) = \mu(u_3u_5) = 0.2$, $\mu(u_1u_4) = 0.4$, and $\mu(u_2u_4) = 0.3$, $\mu(u_3u_4) = 0.1$, and $\mu(u_2u_3) = 0.6$ (Fig. 14.6).

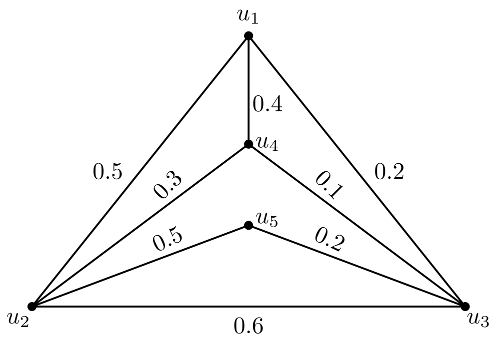

FIGURE 14.6

Fuzzy graph G with a g-cyclic cutvertex u_4.

Here $C^g(G) = 0.3$ and the cycle $u_1u_2u_4u_1$ is of highest strength. Vertices u_1, u_2, and u_4 are the g-cyclic cutvertices of G. Note that the vertex u_4 is a fuzzy end vertex since u_4 has exactly one strong neighbor u_1.

Theorem 14.3.11. *There always exist g-cyclic end vertices in a complete fuzzy graph.*

Proof. Let $G = (\sigma, \mu)$ be a complete fuzzy graph with $|\sigma^*| = n$ where $\sigma^* = \{u_1, u_2, ..., u_n\}$. Let $m_i \in (0,1]$ be such that $\sigma(u_i) = m_i$, $1 \leq i \leq n$, and m_i's are in the increasing order of vertex strength. That is, u_1 is the vertex of least strength. We consider two cases:

Case 1. G contains g-cyclic cutvertices.

Suppose G contain g-cyclic cutvertices. Then $m_{n-3} < m_{n-2}$. In that case, the cycle $u_{n-2}u_{n-1}u_n u_{n-2}$ will have the maximum strength and u_{n-2}, u_{n-1}, and u_n are the g-cyclic cutvertices of G. In fact they are the only g-cyclic cutvertices since m_i's are in the ascending order with $m_{n-3} < m_{n-2}$. Being a CFG, all vertices of G lie on some of its cycles. In particular, the remaining $n-3$ vertices $u_1, u_2, ..., u_{n-3}$ that are not g-cyclic cutvertices lie on some cycles in G. Hence $u_1, u_2, ..., u_{n-3}$ are g-cyclic end vertices of G.

Case 2. G does not contain g-cyclic cutvertices.

Since all vertices of G lie on some cycles, they are precisely the g-cyclic end vertices as none of them are g-cyclic cutvertices of G.

Thus, in both cases, a complete fuzzy graph always contains g-cyclic end vertices. ■

14.4 Cyclically stable fuzzy graphs

A cyclically stable fuzzy graph is one without any g-cyclic cutvertices and g-cyclic bridges. Obviously, a complete fuzzy graph that is cyclically balanced is cyclically stable, and vice versa. Thus we have the following result.

Theorem 14.4.1. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph with $\sigma^* = \{u_1, u_2, ..., u_n\}$, $n \geq 4$. Choose $m_i \in (0, 1]$ such that $\sigma(u_i) = m_i$, $1 \leq i \leq n$, and $m_1 \leq m_2 \leq ... \leq m_n$. Then G is cyclically stable if and only if $m_{n-3} = m_{n-2}$.*

Proof. Let $u_1, u_2, ..., u_n \in \sigma^*$ be such that $\sigma(u_i) = m_i$, $1 \leq i \leq n$ with $m_1 \leq m_2 \leq ... \leq m_n$.

Suppose that G is cyclically stable. We need to prove $m_{n-3} = m_{n-2}$. Suppose not. Then $m_{n-3} < m_{n-2}$. Clearly, the cycle $u_{n-2}u_{n-1}u_n u_{n-2}$ is of maximum strength since $m_{n-3} < m_{n-2}$ and m_i's are in the increasing order of vertex strength. Hence, removal of any of the three vertices u_n, u_{n-1}, and u_{n-2} reduces the g-cycle connectivity of G. Thus u_n, u_{n-1}, and u_{n-2} are the g-cyclic cutvertices of G, which is a contradiction to the fact that G is cyclically stable.

Conversely, suppose that $m_{n-3} = m_{n-2}$. Then the cycles $C' = u_n u_{n-1} u_{n-2} u_n$ and $C'' = u_n u_{n-1} u_{n-3} u_n$ have the same strength and hence removal of u_{n-2}, u_{n-1}, or u_n will not reduce the g-cycle connectivity of G. Therefore there does not exist g-cyclic cutvertex in G. That is, G is cyclically stable. ■

Corollary 14.4.2. *A complete fuzzy graph $G = (\sigma, \mu)$ with $|\sigma^*| \geq 4$ is cyclically stable if there exists a subgraph K_4 of G, where each cycle has the same maximal strength.*

Proof. Consider a complete fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{u_1, u_2, ..., u_n\}$. Let $m_i \in (0,1]$ be such that $\sigma(u_i) = m_i$, $1 \leq i \leq n$, and $m_1 \leq m_2 \leq ... \leq m_n$. A subgraph K_4 of G in which every cycle possesses equal maximum strength occurs only when $m_{n-3} = m_{n-2}$. Then by Theorem 14.4.1, G is cyclically stable. ■

Remark. For a complete fuzzy graph $G = (\sigma, \mu)$, let $x \in \sigma^*$ be such that $d_s(x) = \Delta_s(G)$. Let C be a cycle of maximum strength among all cycles containing x. Suppose $\sigma^* = \{u_1, u_2, ..., u_n\}$ be such that $\sigma(u_1) \leq \sigma(u_2) \leq ... \leq \sigma(u_n)$. Then it can be shown easily that x lies in the strong cycle $u_n u_{n-1} u_{n-2} u_n$. Clearly, this cycle is of maximum strength. Hence, $C = u_n u_{n-1} u_{n-2} u_n$. Since $d_s(x) = \Delta_s(G)$, it is clear that x is of maximum vertex strength. That is, $x = u_n$. Hence, $S(C) = C^g(G)$.

The following result shows the existence of cyclically stable fuzzy graphs with $|\sigma^*| \geq 6$.

Theorem 14.4.3. *Let $G = (\sigma, \mu)$ be a fuzzy graph with $|\sigma^*| \geq 6$. Then G is cyclically stable if it satisfies the following:*

1. *There exist cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $S(\mathcal{C}_1) = S(\mathcal{C}_2) = C^g(G)$.*
2. *All the cycles with strength equal to $C^g(G)$ are disjoint.*

Conversely, if a fuzzy graph is cyclically stable, then there exists at least two disjoint cycles with strength equal to $C^g(G)$.

Proof. Consider a fuzzy graph G with $|\sigma^*| \geq 6$ satisfying *1* and *2*. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two disjoint cycles with $S(\mathcal{C}_1) = S(\mathcal{C}_2) = C^g(G)$. Let $r \in \sigma^*$.

Case 1. $u \notin V(\mathcal{C}_1 \cup \mathcal{C}_2)$

Suppose that u is a vertex not in $\mathcal{C}_1$ and $\mathcal{C}_2$. Then removal of u will not influence the g-cycle connectivity of $G - \{u\}$.

Case 2. $u \in V(\mathcal{C}_1)$ or $u \in V(\mathcal{C}_2)$

Suppose that $u \in V(\mathcal{C}_1)$. Then deletion of u will not alter the g-cycle connectivity of the graph as there exists a different cycle $\mathcal{C}_2$ with strength same as $C^g(G)$.

Similar is the case when $u \in V(\mathcal{C}_2)$.

Hence, u is not a g-cyclic cutvertex. Next we prove there are no g-cyclic bridges in G.

Let $uv \in \mu^*$. Suppose uv is not an edge of $\mathcal{C}_1$ and $\mathcal{C}_2$. That is, $uv \notin E(\mathcal{C}_1 \cup \mathcal{C}_2)$. Then removal of uv will not affect the g-cycle connectivity of the graph $G - \{us\}$. Similarly if $uv \in E(\mathcal{C}_1)$ or $rs \in E(\mathcal{C}_2)$, the removal of the edge uv will not reduce $C^g(G)$ as there always exists another cycle with strength same as the g-cycle connectivity of G. Hence uv is not a g-cyclic bridge of G. Thus G is cyclically stable.

Conversely, suppose that G is cyclically stable. That is, G has no g-cyclic cutvertices and g-cyclic bridges. Suppose that there exists exactly one cycle C of maximum strength. Then clearly all vertices and edges of C are g-cyclic cutvertices and g-cyclic bridges respectively, contradicting the fact that G is cyclically stable. Hence, there exists at least two cycles with strength equal to $C^g(G)$. Let C_1 and C_2 be two cycles of equal strength with strength same as $C^g(G)$. Now it remains to show that C_1 and C_2 are disjoint. Suppose not. Let z be a vertex common to both the cycles C_1 and C_2. Removing the vertex z breaks C_1 and C_2 and hence reduces the g-cycle connectivity of G. This leads to a contradiction since G is cyclically stable. Hence C_1 and C_2 are disjoint. ∎

Next, we analyze the existence of cyclically stable fuzzy graph G with $|\sigma^*| = 4$ and 5. Note that the number of edges incident to a vertex u in G is denoted by the symbol $d(r)$.

Theorem 14.4.4. *A fuzzy graph G with $|\sigma^*| = 4$ is cyclically stable if and only if there exists at least four cycles of maximum strength in G.*

Proof. Suppose G is cyclically stable. We need to prove G contains at least four cycles of maximum strength.

Suppose G has exactly one cycle of maximum strength. Then all its vertices and edges are g-cyclic cutvertices and g-cyclic bridges respectively.

Assume G have two cycles of maximum strength as illustrated below (Fig. 14.7). Then G, being a fuzzy graph with 4 vertices, any pair of cycles in G will have at least one vertex in common. In particular, the pair of cycles with maximum strength will have at least one common vertex and removal of this vertex will reduce the g-cycle connectivity of the graph.

FIGURE 14.7

Fuzzy graph with two cycles of maximum strength.

G with exactly 3 cycles of maximum strength is illustrated in Fig. 14.8. Here there are two vertices, namely b and d, that are g-cyclic cutvertices.

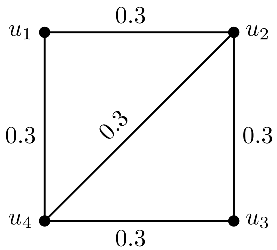

FIGURE 14.8

Fuzzy graph with exactly three cycles of maximum strength.

Thus, we can conclude that G is not cyclically stable if there are no more than 3 cycles of maximum strength.

Now consider a fuzzy graph G as illustrated below (Fig. 14.9). Since G is cyclically stable, removal of any vertex or edge of G will not reduce $C^g(G)$. If that is the case, then the only possibility for G to be cyclically stable is when G contains at least four cycles of maximum strength, since then no vertex or edge will be a g-cyclic cutvertex or g-cyclic bridge.

Conversely, suppose that G has at least four cycles of maximum strength. Let $G = (\sigma, \mu)$ be such that $\sigma^* = \{u_1, u_2, u_3, u_4\}$ and $\mu^* = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_1u_3, u_2u_4\}$ (Fig. 14.9).

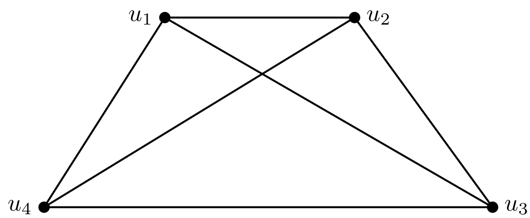

FIGURE 14.9

Fuzzy graph with at least four cycles of maximum strength.

There are two possibilities to consider,

First, Let $C_1 = u_1u_2u_4u_1$, $C_2 = u_1u_2u_3u_1$, $C_3 = u_1u_4u_3u_1$ and $C_4 = u_2u_4u_3u_2$ be four cycles of maximum strength. Then the deletion of any of the vertices or edges of G will not alter $C^g(G)$ and hence G is cyclically stable.

Now let $C_1 = u_1u_2u_4u_1, C_2 = u_1u_2u_3u_1, C_3 = u_1u_4u_3u_1,$ and $C_4 = u_1u_2u_3u_4u_1$ be the four cycles of maximum strength. Consider the cycle $C' = u_4u_3u_2u_4$. If $S(C') < S(C_1)$, then edges of C' will have weight less than $S(C_1)$. This will affect the strength of other cycles in G and hence is a contradiction to our assumption that C_1, C_2, C_3, and C_4 are cycles of maximum strength. Thus no vertices or edges of G are g-cyclic cutvertices or bridges. Clearly $S(C')$ is not greater than $S(C_i)$, $i = 1, 2, 3, 4$, since C_i's are chosen as cycles of maximum strength.

Hence in both possibilities, G is cyclically stable. ■

Theorem 14.4.5. *A fuzzy graph $G = (\sigma, \mu)$ with $|\sigma^*| = 5$ is cyclically stable if it satisfies the following conditions:*

1. *G has at least two vertices r and s with $d(r) = d(s) = 4$ and at most one vertex j with $d(j) \leq 2$.*
2. *Every cycle in G is of equal strength.*

Proof. Suppose that G satisfies conditions *1* and *2*. Let $C^g(G) = p$. Then $s(C) = p$ for all cycles C in G. Since G satisfies 2, it is clear that any vertex u removed from G leaves behind at least one cycle in $G - u$ and hence GCC of the graph is unaltered. Thus no vertex or edge of G is a g-cyclic cutvertex or g-cyclic bridge, respectively, and therefore G is cyclically stable. ■

The converse of Theorem 14.4.5 need not be true. For instance, consider $G = (\sigma, \mu)$ with $\sigma^* = \{u_1, u_2, u_3, u_4, u_5\}$, $\mu(u_1u_2) = \mu(u_4u_5) = 0.6$, $\mu(u_2u_3) = \mu(u_2u_4) = \mu(u_1u_4) = 0.5$, $\mu(u_1u_5) = 0.3$, $\mu(u_1u_3) = \mu(u_3u_4) = 0.7$, and $\mu(u_2u_5) =$

0.8 (Fig. 14.10). Clearly, G is cyclically stable with $C^g(G) = 0.5$. Note that there are cycles of strength other than $C^g(G)$ in G, which violates condition *2* of Theorem 14.4.5, even though G meets condition *1*.

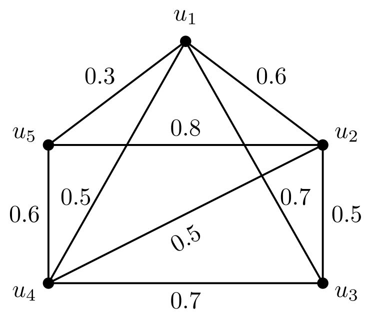

FIGURE 14.10

Fuzzy graph not meeting condition *2* of Theorem 14.4.5.

Both conditions *1* and *2* are necessary for the cyclic stability of G described in Theorem 14.4.5. For example, consider the fuzzy graph G given in Fig. 14.10. G satisfies *1* but not *2* since there are cycles in G of strength different from 0.4, which is the g-cycle connectivity of G. Note that the vertex k is a g-cyclic cutvertex of G and hence G is not cyclically stable.

Fig. 14.11 shows a fuzzy graph G' that contains cycles with the same strength. Let a be the only vertex of G with $d(a) = 4$. That is, G' meets condition *2* but not *1* of Theorem 14.4.5 (Fig. 14.12). It is evident that G' is not cyclically stable as a is a g-cyclic cutvertex of G'.

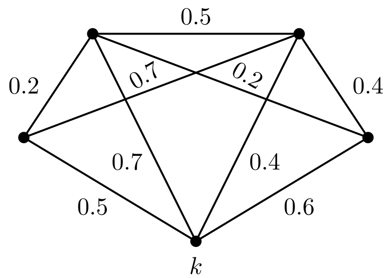

FIGURE 14.11

G satisfying *1* but not *2*.

Theorem 14.4.6. *There always exists a connected cyclically stable fuzzy graph G for $|\sigma^*| \geq 4$.*

Proof. Cyclically stable fuzzy graphs with four and five vertices are given below (Fig. 14.13).

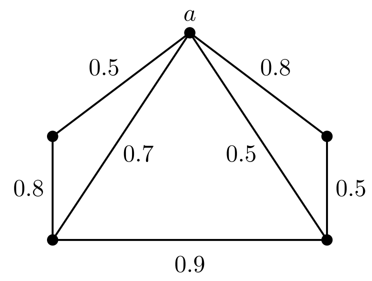

FIGURE 14.12

G' satisfying *2* but not *1*.

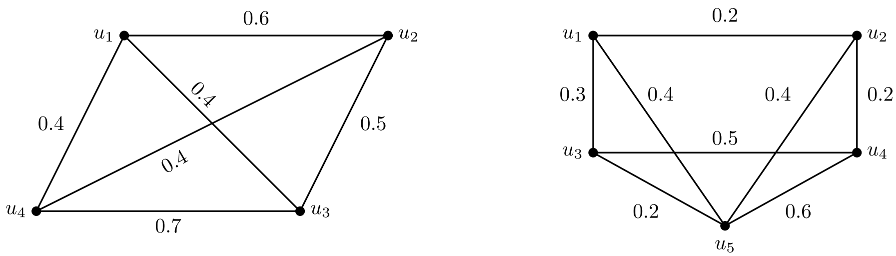

FIGURE 14.13

Cyclically stable fuzzy graphs with four and five vertices.

It is left to construct cyclically stable fuzzy graphs with $|\sigma^*| \geq 6$. The result is proved using induction on $|\sigma^*|$.

Consider a fuzzy graph G with $|\sigma^*| = 6$. Let $u_1, u_2, ..., u_6$ be vertices of G. Construct cycles $C_1 = u_1u_2u_3u_1$ and $C_2 = u_4u_5u_6u_4$ such that they are disjoint with maximum strength. Create edges by joining each vertex in C_1 to every vertex in C_2, thus forming a complete graph (Fig. 14.14). In the complete graph constructed, deletion of a vertex or an edge will not reduce the g-cycle connectivity of G. Hence, the graph thus obtained is a cyclically stable fuzzy graph.

Assume that there is a connected cyclically stable fuzzy graph for $|\sigma^*| = m$. Let G^m be the cyclically stable fuzzy graph with m vertices. Then there exist two cycles that are disjoint and of maximum strength in G^m.

Obtain a fuzzy graph G^{m+1} by adding a vertex, say x, to G^m. Form a complete graph by joining each vertex of G^m to x and assign weights to all newly connected edges in such a way that the weight does not exceed the g-cycle connectivity of G^m. Clearly, on removing the vertex x, g-cycle connectivity of G^m remains the same and also by assumption, no vertex or edge of G^m is a g-cyclic cutvertex or g-cyclic bridge. Hence G^{m+1} is a cyclically stable fuzzy graph. ■

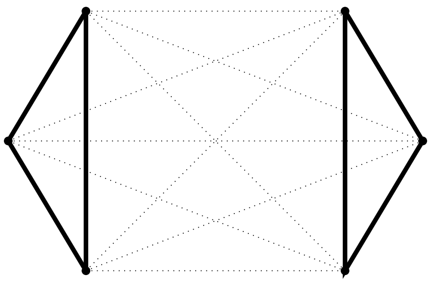

FIGURE 14.14

Construction of cyclically stable fuzzy graph with $|\sigma^*| = 6$.

14.5 Algorithm

This section tries to develop a fuzzy graph clustering algorithm by utilizing the connectivity notions discussed in the previous sections. Clustering can be accomplished by a variety of algorithms and the algorithms for fuzzy graph clustering proposed in the papers [63], [67], [100], and [126] provide the motivation for this section. We present a few definitions that are required in the procedure to obtain a better insight.

Definition 14.5.1. A connected fuzzy graph G is $\mathbf{g^m}$**-cyclic vertex connected** if $\kappa^g(G) \geq m$ and G is called $\mathbf{g^m}$**-cyclic edge connected** if $\overline{\kappa^g}(G) \geq m$.

That is, G is g^m-cyclic vertex connected if there is no g-cyclic vertex cut W with $S^g(W) < m$ and G is g^m-cyclic edge connected if there is no g-cyclic edge cut E with $\overline{S^g}(E) < m$.

Definition 14.5.2. A maximal g^m-cyclic edge connected fuzzy subgraph of G is called a $\mathbf{g^m}$**-cyclic edge component** of G.

Alternatively, a g^m-cyclic edge component of G is a fuzzy subgraph H of G induced by a set of vertices of G in such a way that $\overline{\kappa^g}(H) = m$.

Definition 14.5.3. A set $\tilde{C}$ of vertices of G is called **cyclic cluster of level** $\boldsymbol{m}$ if the fuzzy subgraph of G induced by vertices of $\tilde{C}$ is a g^m-cyclic edge component of G.

A vertex or an edge of a fuzzy graph G is called an element of G.

Definition 14.5.4. **Cyclic cohesiveness** of an element l of a fuzzy graph G denoted by $h_c(l)$ is defined as

$$h_c(l) = \begin{cases} \vee \{C^g_{G_i}(l) : \ G_i' s \ \textit{are the fuzzy subgraphs of G containing l}\}, \textit{if l is an edge.} \\ 0, \ \textit{if l is a vertex.} \end{cases}$$

Definition 14.5.5. Consider a fuzzy graph G. The **cyclic cohesive matrix** L_c of G is given by $L_c = (u_{ij})$ where $u_{ij} = h_c(u_i u_j)$, if i $\neq$ j and $h_c(u_i)$, if i = j.

A vertex u belongs to a cyclic cluster of level m if it is contained in a g^m-cyclic edge component of G. Thus, by finding the clusters of G one can easily identify the

g^m-cyclic edge components of G. The method of finding g^m-cyclic edge components and thus obtaining cyclic clusters in G using g-cyclic edge connectivity $\overline{\kappa^g}$ is known as g^m-cyclic edge connectivity process.

Algorithm

14.5.1 g^m-Cyclic edge connectivity process

Step 1: Generate the Cyclic Cohesive matrix L_c of G.

Step 2: Obtain the m - threshold graph G^m of L_c.

Step 3: The maximal complete subgraphs of G^m represent the g^m-cyclic edge components of G.

14.5.2 Illustration

Consider the fuzzy graph representation of a triangular grid network given in Fig. 14.5. Let H denote the matrix illustration of the fuzzy graph with 10 vertices as given below.

$$H = \begin{bmatrix} 0 & 0.1 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0.15 & 0.2 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.15 & 0 & 0 & 0.35 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.27 & 0 & 0.3 & 0.45 & 0 & 0 \\ 0 & 0.25 & 0.35 & 0.27 & 0 & 0.37 & 0 & 0.55 & 0.65 & 0 \\ 0 & 0 & 0.5 & 0 & 0.37 & 0 & 0 & 0 & 0.75 & 0.6 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.45 & 0.55 & 0 & 0.7 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.65 & 0.75 & 0 & 0.8 & 0 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0.9 & 0 \end{bmatrix}$$

The cyclic cohesive matrix L_c of H is

$$L_c = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0 & 0.27 & 0.35 & 0.35 & 0.27 & 0.35 & 0.35 & 0.35 \\ 0.1 & 0.2 & 0.27 & 0 & 0.27 & 0.27 & 0.3 & 0.3 & 0.27 & 0.27 \\ 0.1 & 0.2 & 0.35 & 0.27 & 0 & 0.37 & 0.27 & 0.55 & 0.55 & 0.37 \\ 0.1 & 0.2 & 0.35 & 0.27 & 0.37 & 0 & 0.27 & 0.37 & 0.6 & 0.6 \\ 0.1 & 0.2 & 0.27 & 0.3 & 0.27 & 0.27 & 0 & 0.3 & 0.27 & 0.27 \\ 0.1 & 0.2 & 0.35 & 0.3 & 0.55 & 0.37 & 0.3 & 0 & 0.55 & 0.37 \\ 0.1 & 0.2 & 0.35 & 0.27 & 0.55 & 0.6 & 0.27 & 0.55 & 0 & 0.6 \\ 0.1 & 0.2 & 0.35 & 0.27 & 0.37 & 0.6 & 0.27 & 0.37 & 0.6 & 0 \end{bmatrix}$$

Generating a cyclic cohesive matrix eases the construction of a threshold graph. We can easily obtain the threshold graph H^m from L_c for any $m \in (0, \infty)$. The maximal complete subgraphs of H^m are the g^m-cyclic edge components of H. Cyclic clusters of level m are formed by the vertices in these components. For instance, let $m = 0.3$. The threshold graph $H^{0.3}$ is

$$H^{0.3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Cyclic clusters of level 0.3 derived from $H^{0.3}$ are $\{r, s, t, u, v, w, x, y\}$, $\{p\}$, $\{q\}$.

Similarly, cyclic clusters of all levels derived from the corresponding threshold matrices are listed below:

Level	Cyclic clusters
$(1, \infty)$	$\{p\}, \{q\}, \{r\}, \{s\}, \{t\}, \{u\}, \{v\}, \{w\}, \{x\}, \{y\}$
$(0.6, 1]$	$\{p\}, \{q\}, \{r\}, \{s\}, \{t\}, \{u\}, \{v\}, \{w\}, \{x\}, \{y\}$
$(0.55, 0.6]$	$\{u, x, y\}, \{p\}, \{q\}, \{r\}, \{s\}, \{t\}, \{v\}, \{w\}$
$(0.35, 0.55]$	$\{t, u, w, x, y\}, \{p\}, \{q\}, \{r\}, \{s\}, \{v\}$
$(0.3, 0.35]$	$\{r, t, u, w, x, y\}, \{p\}, \{q\}, \{s\}, \{v\}$
$(0.2, 0.3]$	$\{r, s, t, u, v, w, x, y\}, \{p\}, \{q\}$
$(0.1, 02]$	$\{q, r, s, t, u, v, w, x, y\}, \{p\}$
$(0, 0.1]$	$\{p, q, r, s, t, u, v, w, x, y\}$

14.6 Application

This section showcases the utility of the concepts discussed in the previous sections by portraying two important concerns of the modern world.

Application 1: Youth mental illness and homelessness

In [128], it is stated that an estimated 4.2 million youth and young adults experience homelessness, of which 700, 000 are unaccompanied minors, meaning they are not part of a family or accompanied by a parent or guardian. On any given night, ap-

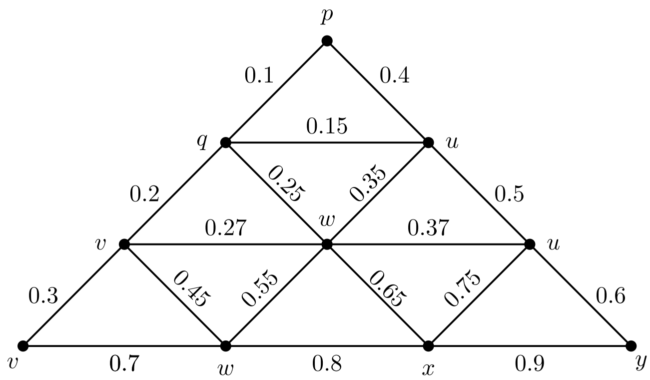

FIGURE 14.15

Triangular Grid Network.

proximately 41, 00 unaccompanied youth ages 13–15 experience homelessness. One in 10 young adults 18-15, and at least one in 30 adolescents ages 13–17, experience some form of homelessness unaccompanied by a parent or guardian (Fig. 14.15).

It is stated in [21] that mental health is an important part of children's overall health and well-being. Mental health includes children's mental, emotional, and behavioral well-being. It affects how children think, feel, and act. It also plays a role in how children handle stress, relate to others, and make healthy choices. ADHD (Attention Deficit Hyperactivity Disorder), anxiety problems, behavior problems, and depression are the most commonly diagnosed mental disorder in children. Estimates for ever having a diagnosis among children aged 3–17 years, in 2016–2019, are given in [21] to be ADHD 9.8%, (approximately 6.0 million) Anxiety 9.4%, (approximately 5.8 million), Behavior problems 8.9%, (approximately 5.5 million), Depression 4.4% (approximately 2.7 million). This data is modeled using fuzzy graphs where the mental disorders mentioned above form the vertices of the graph. These vertices are assigned a membership value of 1, indicating full membership. Here, we consider a anxiety, b behavior, d depression, m mental disorder, and c development disorder as vertices of a fuzzy graph, G.

Some of these conditions commonly occur together, forming an edge in the fuzzy graph G. For example, if children suffering from depression (d) also experience anxiety (a), then there is an edge ad in the corresponding fuzzy graph. Membership values are assigned to the edges based on the approximations in [21], with the decimal equivalent of the percentage approximation serving as the degree of membership for the edges. For instance, if 50% of children with depression also experience anxiety, then the edge ad is assigned a weight of 0.5. For convenience, the fuzzy graph is considered undirected.

Edge	ab	ad	ac	am	bc	bd	bm	cd	cm	dm
Weight	0.5	0.5	0.4	0.4	0.4	0.35	0.45	0.2	0.3	0.35

We see that if we delete edge ab, i.e., assign the weight 0 to ab, then the resulting fuzzy graph H is such that $C^g(H) = 0.35$ while $C^g(G) = 0.4$ (Fig. 14.16). Thus ab is g-cyclic bridge and so a and b are g-cyclic cut vertices. From this information, we can conclude that the most effective way to reduce mental illness in youth (including toxic stress) is concentrating on the anxiety, behavior connection.

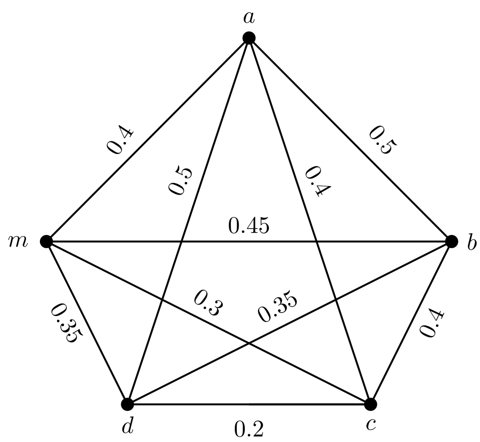

FIGURE 14.16

Fuzzy graph illustration to Application 1.

Application 2: Country vulnerability to human trafficking

Consider a directed graph $\vec{G} = (V, \vec{E})$, where V is a set of vertices and $\vec{E} \subseteq V \times V$ is a set of directed edges. One can learn properties of the directed graph by ignoring the direction. In fact, in certain cases, the direction is not important. We illustrate this for fuzzy-directed graphs related to human trafficking. Suppose $\vec{G} = (V, \vec{E})$ has the property that if $(u, v) \in \vec{E}$, then $(v, u) \notin \vec{E}$. Let (V, E) be the graph defined by letting $E = \{\{u, v\} | (u, v) \in \vec{E}\}$. Define $\vec{f} : \vec{E} \to E$ by $\vec{f}((u, v)) = \{u, v\}$. Then $\vec{f}$ is a one-to-one function of $\vec{E}$ onto E which preserves adjacency, i.e., $(u, v) \in \vec{E}$ if and only if $\vec{f}((u, v)) \in E$. This leads to an isomorphism between $\vec{G}$ and G. For fuzzy directed graphs, we assign the same weight to $\{u, v\}$ that is assigned to (u, v). Of course, these ideas can be generalized; a project to be dealt with later.

We concentrate on trafficking routes from countries to South America to the United States through Mexico. We are interested in the vulnerability of countries to human trafficking. We define the flow from a country to another as a combination of the origin flow and the transit flow through the country, and the flow into a country as a combination of the transit flow through it and the destination flow into it. This combination is accomplished by using the algebraic conorm of these values. That is, for all p, q in [0,1], algebraic conorm of p and q, denoted as $p \circ q$ is the value $p + q - pq$. If the trafficking route between two countries is active, the weight provided to the edge thus formed is the algebraic conorm of the vulnerabilities of the two countries. In this context, the direction of trafficking is not an essential criterion; therefore the modeled fuzzy graph is undirected to simplify the analysis. Vulnerability estimates how vulnerable people in a country are to human trafficking. Very low (VL) vulnerability is assigned a value of 0.1, 0.3 for low (L), 0.5 for medium (M), 0.7 for high (H), and 0.9 for very high (VH) [70] B1987. We have the following edges and weights given in the following table, [77,115]. They were determined from [89,121]. We combine the source countries of the routes, China, India, Ethiopia, Somalia, and Nigeria, into one Source. Since there is flow to the United States from only one country, namely Mexico, we do not consider the United States.

In the following table, S denotes Source, C denotes Columbia, G denotes Guatemala, M denotes Mexico, SA denotes South Africa, B denotes Brazil, E denotes Ecuador, UAE denotes United Arab Emirates, R denotes Russia, Cu denotes Cuba, and SP denotes Spain.

Country	S	C	G	SA	B	E	M	UAE	R	Cu	Sp
Vulnerability	H	L	M	H	VL	M	L	L	L	M	VL

Accordingly, the edge weights are approximated as follows:

Edge	S, C	C, G	G, M	S, SA	SA, B	B, E	E, M	S, UAE
Weight	0.79	0.65	0.65	0.91	0.73	0.55	0.65	0.79

Edge	UAE, R	R, Cu	Cu, C	S, Sp	Sp, Cu	S, G	C, M
Weight	0.51	0.65	0.65	0.73	0.55	0.85	0.51

Note that the vertices are assigned a membership value of 1, indicating full membership. We wish to examine where we can reduce the vulnerability of the fuzzy directed graph. We can see that the direction of the edge is not needed in this particular objective.

The strongest cycle containing edge $SGuatemala$ is of strength 0.65. It is obtained from the cycle $S, Guatemala, Columbia, S$ (Fig. 14.17). Thus $C^g(G) = 0.65$. If we delete $SGuatemala$ to obtain the fuzzy subgraph H, we have $C^g(H) = 0.55$. The conclusion drawn from these findings is that taking measures to limit human trafficking from source nations to Guatemala can have a positive impact on lowering the global rate of human trafficking. The numerical values assigned as the weights of the directed edges offer a concrete measure of how these relationships influence the network. This information provides valuable insights that can guide the development of effective strategies to tackle human trafficking on a broader scale.

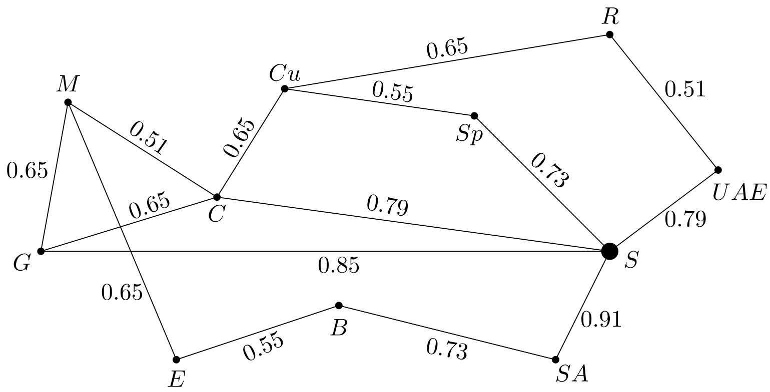

FIGURE 14.17

Fuzzy graph showing country vulnerability to human trafficking.

14.7 Exercises

1. Calculate the generalized cycle connectivity of $G = (\sigma, \mu)$ with vertex set $\sigma^* = \{u_1, u_2, u_3, u_4, u_5\}$, $\mu(u_1u_2) = 0.4$, $\mu(u_1u_3) = 0.5$, $\mu(u_2u_3) = 0.2$, $\mu(u_2u_4) = 0.7$, $\mu(u_3u_4) = 0.6$, $\mu(u_4u_5) = 0.9$, $\mu(u_2u_5) = 0.2$, and $\mu(u_3u_5) = 0.1$.

2. Identify the g-cyclic cut vertices and g-cyclic bridges of $G = (\sigma, \mu)$ with $\sigma^* = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mu(u_1u_2) = 0.7$, $\mu(u_1u_3) = 1$, $\mu(u_1u_5) = \mu(u_4u_5) = 0.4$, $\mu(u_1u_6) = \mu(u_2u_3) = 0.8$, $\mu(u_2u_4) = 0.3$, $\mu(u_2u_6) = 0.9$, $\mu(u_3u_4) = \mu(u_3u_5) = 0.6$, $\mu(u_4u_6) = 0.3$, and $\mu(u_5u_6) = 0.3$.

3. Consider a fuzzy graph $G = (\sigma, \mu)$. Let $H = (\tau, \nu)$ be a partial fuzzy subgraph of G. Then prove that $\kappa^g(H) \leq \kappa^g(G)$.

CHAPTER

Fuzzy vertex and edge connectivity[☆] 15

Fuzzy vertex and edge connectivity play crucial roles in the dynamics of interconnection networks. Average fuzzy edge connectivity (AFEC) is a better tool to compare the capacity of large-scale networks. Also, the stability and performance of modern networks depend largely on the connectivity properties of the graph model. Average edge connectivity is a means to comprehend the collective relationship intensity within a fuzzy graph. This chapter depends upon [107].

15.1 Average fuzzy vertex connectivity

We discuss average fuzzy vertex connectivity *(AFVC)* of fuzzy graphs in this section, several different fuzzy graph structures are considered.

Lemma 15.1.1. *All β-strong edges of a fuzzy cycle have the same strength.*

Proof. Since fuzzy cycles have no δ-edges, every edge in a fuzzy cycle is strong. The weakest edges are the β-strong edges because fuzzy cycles have multiple weakest edges. Let xy be a weakest edge in the fuzzy graph G. Suppose there exists a β-strong edge uv whose strength exceeds $\mu(xy)$. Then, there exists a strongest $u-v$ path P in G not containing uv but containing xy. Then, $CONN_{G-uv}(u,v) = \mu(xy)$, since xy is the weakest edge in G, this contradicts the fact that uv is a β-strong edge. ■

Theorem 15.1.2. *For a saturated fuzzy cycle G with weight of every α-strong edge is υ and weight of every β-strong edge is ι, then we have, $\sum_{u,v\in G} \kappa_G(u,v) = \frac{n}{2}\upsilon + \iota[n(n-2)]$, where n is the number of vertices.*

Proof. Theorem 4.3 in [2] states that the α-strong and β-strong edges alternately arise and $n = 2k$ for an integer k when G is a saturated fuzzy cycle with n vertices. Let $\{v_1, v_2, \ldots, v_n\}$ be vertices in G, and $\{v_1v_2, v_3v_4, \ldots, v_{n-1}v_n\}$ be α-strong edges in G. Considering $v_1 - v_i$ paths for $i = 2, 3, \ldots, n$, only $v_1 - v_2$ path has strength υ and all other $v_1 - v_i$, $i = 3, 4, \ldots, n$, paths have strength ι as it contains β-strong

[☆] This book has a companion website hosting complementary materials. Visit this URL to access it: https://www.elsevier.com/books-and-journals/book-companion/9780443339493.

Fuzzy Mathematics, Graphs, and Similarity Measures. https://doi.org/10.1016/B978-0-44-333949-3.00023-4

edges. There are precisely two internally disjoint paths between v_1 and v_i for $i = 3, 4, \ldots, n$ because the strength of each $v_1 - v_i$ path for $i = 3, 4, \ldots, n$ is equal to the strength of a β-strong edge. Thus $\sum_{u \in \sigma^* \setminus \{v_1\}} mCONN_G(v_1, u) = \upsilon + 2(n-2)\iota$. Similarly considering $v_2 - v_i$, $i = 3, 4, \ldots n$, paths there is no α-strong edge incident on v_2 other than $v_1 v_2$ and all the $v_2 - v_i$, $i = 3, 4, \ldots, n$ paths have strength ι. Hence, $\sum_{u \in \sigma^* \setminus \{v_1, v_2\}} mCONN_G(v_2, u) = 2(n-2)\iota$. For v_3, only the $v_3 - v_4$ path has strength υ and all other $v_3 - v_i$ path has strength ι for $i = 5, 6, \ldots, n$. Thus

$$\sum_{u \in \sigma^* \setminus \{v_1, v_2, v_3\}} mCONN_G(v_3, u) = \upsilon + 2(n-4)\iota.$$

Correspondingly for v_4,

$$\sum_{u \in \sigma^* \setminus \{v_1, v_2, v_3, v_4\}} mCONN_G(v_4, u) = 2(n-4)\iota,$$

and so on. In general,

$$\sum_{u \in \sigma^* \setminus \{v_1, v_2, \ldots, v_t\}} mCONN_G(v_t, u) = \upsilon + 2(n-(t+1))\iota, \text{ if } t \text{ is odd}$$

and

$$\sum_{u \in \sigma^* \setminus \{v_1, v_2, \ldots, v_t\}} mCONN_G(v_t, u) = 2(n-t)\iota, \text{ if } t \text{ is even.}$$

Hence,

$$\begin{aligned}\sum_{u,v \in G} \kappa_G(u, v) &= \upsilon + 2\iota(n-2) + 2\iota(n-2) + \upsilon + 2\iota(n-4) + 2\iota(n-4) + \cdots \\ &\quad + \upsilon + 2\iota(n-(n-2)) + 2\iota(n-(n-2)) + \upsilon \\ &= \frac{n}{2}\upsilon + 4\iota[(n-2) + (n-4) + \ldots + 2] \\ &= \frac{n}{2}\upsilon + \iota[n(n-2)].\end{aligned}$$

■

Corollary 15.1.3 provides the fuzzy vertex connectivity of saturated fuzzy cycles characterized by α-strong edges with unique strengths.

Corollary 15.1.3. *For a saturated fuzzy cycle C_n with α-strong edges of strengths $\upsilon_1, \upsilon_2, \ldots, \upsilon_{\frac{n}{2}}$, we have* $\sum_{u,v \in C_n} \kappa_{C_n}(u, v) = \upsilon_1 + \upsilon_2 + \cdots + \upsilon_{\frac{n}{2}} + 4\iota[\frac{n(n-1)}{2}]$.

Example 15.1.4. Consider Fig. 15.1, a saturated fuzzy cycle C with α-strong edges v_1v_2, v_3v_4, v_5v_6, v_7v_8 having weights $\mu(v_1v_2) = \upsilon_1 = 0.9$, $\mu(v_3v_4) = \upsilon_2 = 0.7$, $\mu(v_5v_6) = \upsilon_3 = 0.7$, $\mu(v_7v_8) = \upsilon_4 = 0.6$ and β-strong edges having weight $\iota = 0.4$.

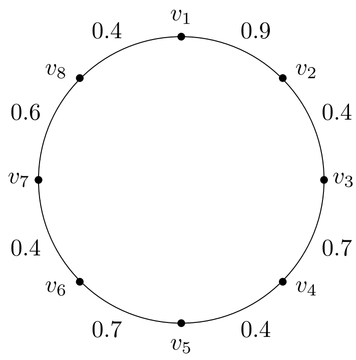

FIGURE 15.1

Saturated fuzzy cycle C in Example 15.1.4.

The edge v_1v_2 being α-strong, we obtain $\kappa_C(v_1, v_2) = \upsilon_1$. All other $v_1 - v_i$ paths for $i = 3, \ldots, 8$ contain β-strong edges of weight ι, and being a cycle there exist exactly two strongest $v_1 - v_i$ paths. As a result, for $i = 3, \ldots, 8$, $\kappa_C(v_1, v_i) = 2\iota$ and $\sum_{i=2}^{8} \kappa_C(v_1, v_i) = \upsilon_1 + 2(6)\iota$. All $v_2 - v_i$ paths for $i = 3, \ldots, 8$ contain β-strong edges and have strength ι. Thus $\sum_{i=3}^{8} \kappa_C(v_2, v_i) = 2(6)\iota$. Similarly,

$$\sum_{i=4}^{8} \kappa_C(v_3, v_i) = \upsilon_2 + 2(4)\iota.$$

$$\sum_{i=5}^{8} \kappa_C(v_4, v_i) = 2(4)\iota.$$

$$\sum_{i=6}^{8} \kappa_C(v_5, v_i) = \upsilon_3 + 2(2)\iota.$$

$$\sum_{i=7}^{8} \kappa_C(v_6, v_i) = 2(2)\iota.$$

$$\kappa_C(v_7, v_8) = \upsilon_4.$$

Consequently,

$$\sum_{u,v \in C} \kappa_C(u, v) = \upsilon_1 + \upsilon_2 + \upsilon_3 + \upsilon_4 + 2(6)\iota + 2(6)\iota + 2(4)\iota + 2(4)\iota + 2(2)\iota + 2(2)\iota$$

$$= 22.1.$$

Then $AFVC$ of C will be $\overline{\kappa}(C) = 1.77$.

Theorem 15.1.5 is a characterization of fuzzy graphs where each pair of vertices is joined by a unique strongest path.

Theorem 15.1.5. *For a connected fuzzy graph $G = (\sigma, \mu)$ and for any $u, v \in \sigma^*$, $\kappa_G(u, v) = CONN_G(u, v)$ if and only if any of the following occurs*

(1) u or v is an endvertex in G

(2) There is a cutvertex $x \in \sigma^*$ that, when removed, reduces connectivity between u and v.

Proof. Let $\kappa_G(u, v) = CONN_G(u, v)$ for vertices u and v in a fuzzy graph G. Then, there exists precisely one vertex-disjoint strongest path from u to v. Let $|\sigma^*| = l$. Clearly, both u and v are end vertices when $n = 2$. Assume $n > 2$. Since there is only one strongest path connecting u and v, removing any vertex from that path reduces the connectivity between u and v. As a result, every vertex in the strongest path that connects u and v will be a cutvertex in G.

Consider, instead, that either (1) or (2) of the conditions occur. First, let v be an end vertex. A unique path will exist between any arbitrary vertex of G and v. Thus $\kappa_G(u, v) = CONN_G(u, v)$, hence condition (1) follows. Second, suppose a cutvertex $x \in \sigma^*$ exists, which, when removed, reduces the connectivity between u and v. Any path between u and v which is strongest will then pass through x. As a result, only one vertex-disjoint strongest $u - v$ path exists, and hence $\kappa_G(u, v) = CONN_G(u, v)$. ■

For a partial fuzzy subgraph H of G, $\overline{\kappa}(H) \leq \overline{\kappa}(G)$ is not generally true. We emphasize that the statement in Theorem 4.22 in [2] may not hold in all cases. Specifically, $\overline{\kappa}(G - uv)$ is not necessarily always less than or equal to $\overline{\kappa}(G)$, and similarly, $\overline{\kappa}(G - d)$ may not always be less than or equal to $\overline{\kappa}(G)$ for a graph $G = (\sigma, \mu)$ and for all $uv \in \mu^*$. This is demonstrated in Examples 15.1.6 and 15.1.7.

Example 15.1.6. Examine Fig. 15.2, the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $\mu(v_1v_2) = 0.6$, $\mu(v_1v_2) = \mu(v_1v_4) = \mu(v_1v_6) = \mu(v_2v_3) = \mu(v_3v_4) = \mu(v_3v_5) = \mu(v_5v_6) = 0.4$. There are 15 pairs of vertices v_iv_j, $1 \leq i <$

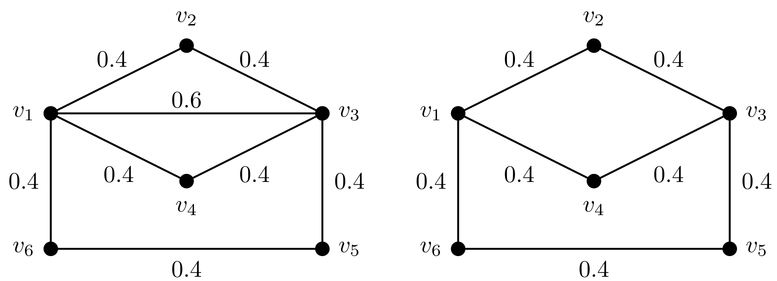

FIGURE 15.2

Fuzzy graph G and $G - v_1v_3$ in Example 15.1.6.

$j \le 6$, where v_1v_3 is an α-strong edge and $\kappa_G(v_1, v_3) = \mu(v_1v_3) = 0.6$. For all other pairs u and v, there are precisely two strongest paths between them with strength 0.4 and $\kappa_G(u, v) = 2CONN_G(u, v) = 0.8$. The AFVC of G is given by

$$\overline{\kappa}(G) = \frac{\sum\limits_{u,v \in G} \kappa_G(u, v)}{\sum\limits_{u,v \in G} CONN_G(u, v)} = \frac{14 \times 2 \times 0.4 + 0.6}{14 \times 0.4 + 0.6} = 1.90.$$

The fuzzy graph $G - v_1v_3$ is obtained by removing the α-strong edge v_1v_3 from G. The connectivity between v_1 and v_3 decreases to 0.4, with three internally disjoint strongest $v_1 - v_3$ paths. The AFVC of $G - v_1v_3$ becomes

$$\overline{\kappa}(G - v_1v_3) = \frac{\sum\limits_{u,v \in G - v_1v_3} \kappa_{G-v_1v_3}(u, v)}{\sum\limits_{u,v \in G - v_1v_3} CONN_{G-v_1v_3}(u, v)} = \frac{14 \times 2 \times 0.4 + 3 \times 0.4}{14 \times 0.4 + 0.4} = 2.06.$$

In this case, we can conclude that, $\overline{\kappa}(G - v_1v_3) \ge \overline{\kappa}(G)$. If we remove any edge v_iv_j other than v_1v_3 from the given graph G, we get $CONN_{G-v_iv_j}(u, v) = CONN_G(u, v)$ and $\sum\limits_{u,v \in G - v_iv_j} \kappa_{G-v_iv_j}(u, v) < \sum\limits_{u,v \in G} \kappa_G(u, v)$ which in turn results in $\overline{\kappa}(G - v_iv_j) \le \overline{\kappa}(G)$. Thus it is not always true that $\overline{\kappa}(G - uv) \le \overline{\kappa}(G)$ for all uv in G.

Using the properties of β-strong and δ edges, we can infer that for any β-strong or δ edge uv,

$$\overline{\kappa}(G - uv) = \frac{\sum\limits_{x,y \in G - uv} \kappa_{G-uv}(x, y)}{\sum\limits_{x,y \in G - uv} CONN_{G-uv}(x, y)} \le \frac{\sum\limits_{x,y \in G} \kappa_G(x, y)}{\sum\limits_{x,y \in G} CONN_G(x, y)} = \overline{\kappa}(G).$$

Now we consider the AFVC of a fuzzy subgraph obtained by the removal of an arbitrary vertex. Consider the example given below.

Example 15.1.7. Consider the fuzzy graph G in Fig. 15.3 and its subgraph obtained from G by removing a vertex. For G, $\sigma^* = \{v_1, v_2, \dots, v_7\}$ with $\mu(v_1v_7) = 0.8$, $\mu(v_1v_2) = 0.6$, $\mu(v_1v_6) = \mu(v_6v_7) = 0.4$, $\mu(v_2v_3) = 0.4$, $\mu(v_2v_4) = 0.3$, $\mu(v_2v_5) = 0.5$, and $\mu(v_2v_6) = 0.2$. There are two vertex-disjoint strongest paths having the same strength of 0.4 for each of the pairs (v_1, v_6) and (v_6, v_7). Every other pair of vertices is connected by a unique strongest path. We obtain the AFVC $\overline{\kappa}(G) = \frac{6.4}{5.6} = 1.14$ by determining the strength of connectivity between each pair of vertices.

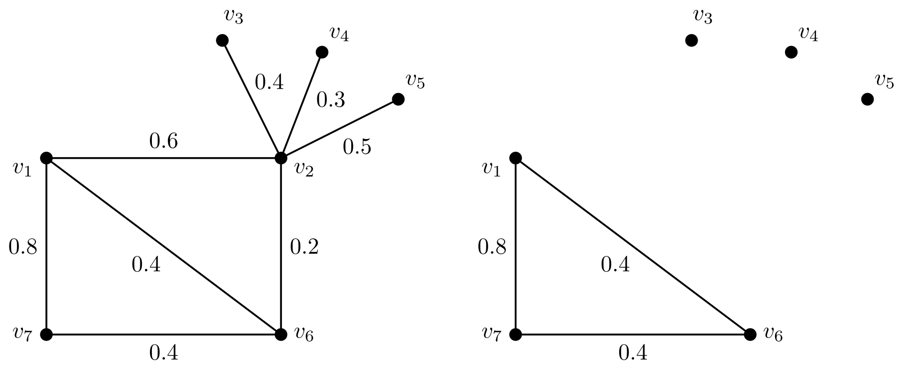

FIGURE 15.3

Fuzzy graph G and $G - v_2$ in Example 15.1.7.

Now, consider $G - v_2$, the fuzzy graph obtained by removing the vertex v_2. There is no change in the strength of connectedness for the pairs (v_1, v_6), (v_1, v_7), and (v_6, v_7) on the removal of v_2. All other pairs of vertices have zero strength of connectedness between them. Consequently, $\overline{\kappa}(G - v_2) = \frac{2.4}{1.6} = 1.5$ is the AFVC of $G - v_2$. The statement that $\overline{\kappa}(G - v) \leq \overline{\kappa}(G)$ for every $v \in \sigma^*$ is therefore not always true.

Theorem 15.1.8. *For a fuzzy graph $G = (\sigma, \mu)$, such that G^*, is a cycle, we have for all $d \in \sigma^*$ and $uv \in \mu^*$,*

(i) $\overline{\kappa}(G - v) \leq \overline{\kappa}(G)$
(ii) $\overline{\kappa}(G - uv) \leq \overline{\kappa}(G)$.

Proof. Suppose G^* is a cycle for the fuzzy graph $G = (\sigma, \mu)$. The fuzzy graph obtained by removing any vertex v is a tree, with each vertex connected to the others by a unique strongest path. Hence,

$$\overline{\kappa}(G - v) = \frac{\sum\limits_{x,y \in G-v} \kappa_{G-v}(x, y)}{\sum\limits_{x,y \in G-v} CONN_{G-v}(x, y)} = \frac{\sum\limits_{x,y \in G-v} CONN_{G-v}(x, y)}{\sum\limits_{x,y \in G-v} CONN_{G-v}(x, y)} = 1.$$

Similar to this, for an edge $uv \in \mu^*$, $G - uv$ is a spanning subgraph which is a tree. Then there exists precisely one strongest path between any two vertices in $G - uv$ and

$$\overline{\kappa}(G - uv) = \frac{\sum\limits_{x,y \in G-uv} \kappa_{G-uv}(x, y)}{\sum\limits_{x,y \in G-uv} CONN_{G-uv}(x, y)} = \frac{\sum\limits_{x,y \in G-uv} CONN_{G-uv}(x, y)}{\sum\limits_{x,y \in G-uv} CONN_{G-uv}(x, y)} = 1.$$

Also, for any non-trivial connected graph G, $\overline{\kappa}(G) \geq 1$. Hence $\overline{\kappa}(G-d) \leq \overline{\kappa}(G)$ and $\overline{\kappa}(G-uv) \leq \overline{\kappa}(G)$. ■

15.2 Average fuzzy edge connectivity

We discuss AFEC in this section. The concepts of AFVC and AFEC differ in the cases with varying numbers of vertex-disjoint strongest paths and edge-disjoint strongest paths. Consider Examples 15.2.2 and 15.2.2.

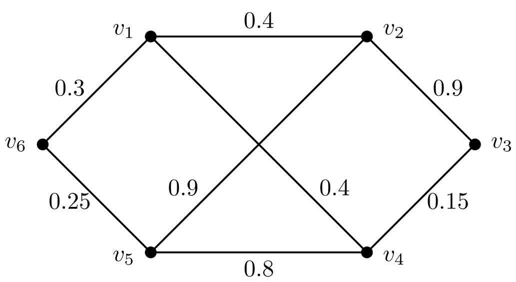

FIGURE 15.4

Graph in Example 15.2.1.

Example 15.2.1. Consider $G=(\sigma,\mu)$ in Fig. 15.4 with $\sigma^*=\{v_1, v_2, \ldots, v_6\}$, $\mu(v_1v_2)=0.4$, $\mu(v_2v_3)=0.9$, $\mu(v_3v_4)=0.15$, $\mu(v_4v_5)=0.8$, $\mu(v_5v_6)=0.25$, $\mu(v_6v_1)=0.2$, $\mu(v_1v_4)=0.4$, and $\mu(v_2v_5)=0.9$. Two internally disjoint strongest paths exist between v_1 and v_4. All other pairs of vertices have exactly one strongest path between them, similar to the case of edge-disjoint strongest paths. So, in this case, there is a likeness between average fuzzy vertex and AFEC.

Example 15.2.2. Let $G=(\sigma,\mu)$ with $\sigma^*=\{v_1, v_2, v_3, v_4, v_5\}$, $\mu(v_3v_4)=\mu(v_4v_5)=0.2$, and $\mu(v_1v_2)=\mu(v_1v_5)=\mu(v_2v_5)=\mu(v_3v_5)=1$ in Fig. 15.5. The strongest edge-disjoint v_2-v_4 paths are $v_2v_5v_4$ and $v_2v_1v_5v_3v_4$. But they are not vertex disjoint. So, here we have more edge-disjoint v_2-v_4 strongest paths than vertex-disjoint v_2-v_4 strongest paths.

This distinction led us to formulate the concept of AFEC. Dissimilarities arise because the edge-disjoint paths may not be internally disjoint. To address this, we introduce the notion of average edge connectivity and find different results related to it.

Definition 15.2.3. In the fuzzy graph $G=(\sigma,\mu)$, let m' signify the number of edge disjoint strongest $u-v$ paths for a pair of vertices (u,v). The pair of vertices u, v is referred to as being *k-edge connected* if $m'CONN_G(u,v) \geq k$. The (u,v)-edge connectivity, or $\lambda_G(u,v)$, of a fuzzy graph $G=(\sigma,\mu)$ is the maximum value of k for which u and v are k-edge connected.

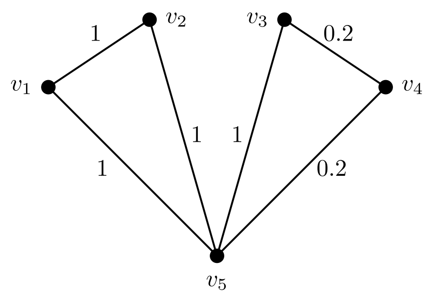

FIGURE 15.5

Fuzzy graph in Example 15.2.2.

Example 15.2.4. All the edges in G, the graph given in Example 15.2.2, are strong. Also, v_1v_2, v_1v_5, v_2v_5, v_3v_4, and v_4v_5 are β-strong edges, each having exactly two edge-disjoint strongest paths. Hence $\lambda_G(v_1, v_2) = 2CONN_G(v_1, v_2) = 2(1) = 2$, $\lambda_G(v_1, v_5) = 2CONN_G(v_1, v_5) = 2(1) = 2$, $\lambda_G(v_2, v_5) = 2CONN_G(v_2, v_5) = 2(1) = 2$, $\lambda_G(v_3, v_4) = 2CONN_G(v_3, v_4) = 2(0.2) = 0.4$, and $\lambda_G(v_4, v_5) = 2CONN_G(v_4, v_5) = 2(0.2) = 0.4$. v_3v_5 is an α-strong edge with unique strongest path and $\lambda_G(v_3v_5) = CONN_G(v_3v_5) = \mu(v_3v_5) = 1$. The pairs v_2v_3 and v_1v_3 have unique edge-disjoint strongest path and $\lambda_G(v_2, v_3) = 1 = \lambda_G(v_1, v_3)$.

The vertices v_1 and v_4 have two edge-disjoint strongest paths $v_1v_5v_4$ and $v_1v_2v_5v_3v_4$ with strength 0.2. Similarly, for the vertices v_2 and v_4, the strongest edge disjoint paths are $v_2v_5v_4$ and $v_2v_5v_3v_4$ of strength 0.2. Thus $\lambda_G(v_1, v_4) = 2(0.2) = 0.4 = \lambda_G(v_2, v_4)$. The total fuzzy edge connectivity is given by $\sum\limits_{u,v} \lambda_G(u, v) = 2 + 2 + 2 + 0.4 + 0.4 + 1 + 1 + 1 + 0.4 + 0.4 = 10.6$.

The AFEC of a fuzzy graph G is the ratio of total fuzzy edge connectivity to the total connectivity of all pairs of vertices in G, that is,

$$\overline{\lambda}(G) = \frac{\sum\limits_{u,v\in G} \lambda_G(u, v)}{\sum\limits_{u,v\in G} CONN_G(u, v)}.$$

The fuzzy graph in Example 15.2.2 has an AFEC of $\overline{\lambda}(G) = \frac{10.6}{6.8} = 1.55$.

Consider a fuzzy graph G. Let m denote the number of vertex-disjoint strongest paths and m' denote the number of edge-disjoint strongest paths for any two vertices $u, v \in \sigma^*$. Clearly, $m \leq m'$ and $mCONN_G(u, v) \leq m'CONN_G(u, v)$ for pairs $u, v \in \sigma^*$, as all vertex-disjoint strongest paths are also edge-disjoint strongest paths. Consequently, $\kappa_G(u, v) \leq \lambda_G(u, v)$ and $\overline{\kappa}(G) \leq \overline{\lambda}(G)$. In Examples 15.2.1 and 15.2.2 of fuzzy graphs, respectively, $\overline{\kappa}(G) = \overline{\lambda}(G)$ and $\overline{\kappa}(G) < \overline{\lambda}(G)$.

Theorem 15.2.5. *If $G = (\sigma, \mu)$ is a fuzzy graph, then $\overline{\lambda}(G) = 0$ if and only if $\overline{\kappa}(G) = 0$.*

Proof. Let G be a fuzzy graph with $\overline{\lambda}(G) = 0$, then clearly through the previous paragraph $\overline{\kappa}(G) = 0$. Conversely, let $\overline{\kappa}(G) = 0$. Then,

$$\frac{\sum\limits_{u,v \in G} \kappa_G(u, v)}{\sum\limits_{u,v \in G} CONN_G(u, v)} = 0,$$

i.e., $\sum\limits_{u,v \in G} \kappa_G(u, v) = 0$. Thus $CONN_G(u, v) = 0$, for each $u, v \in G$ which implies $\lambda_G(u, v) = 0$ for each $u, v \in G$ and in turn $\overline{\lambda}(G) = 0$. ■

It is important to note that for a fuzzy graph G with l vertices, $\overline{\lambda}(G) = 0$ if and only if G is trivial when $l = 1$ and G is totally disconnected when $l > 1$. Now, let us examine the AFEC of some particular fuzzy graphs.

Theorem 15.2.6. *For an edge-disjoint fuzzy graph G, G is a fuzzy forest, then $\overline{\lambda}(G) = 1$ and vice versa.*

Proof. For an edge-disjoint fuzzy graph G, G is a fuzzy forest if and only if there is at most one edge-disjoint strongest path between any two vertices of G if and only if $\lambda_G(u, v) = CONN_G(u, v)$ for each $u, v \in \sigma^*$ if and only if $\sum\limits_{u,v \in G} \lambda_G(u, v) = \sum\limits_{u,v \in G} CONN_G(u, v)$ if and only if $\overline{\lambda}(G) = 1$. ■

Theorem 15.2.7. *If G is a fuzzy cycle, then $\overline{\lambda}(G) \leq 2$.*

Proof. Each fuzzy cycle G contains at least two weak edges. Being a cycle, exactly two edge-disjoint paths exist between any two vertices in G. Take two vertices, u and v. Then there exist two paths, say P_1 and P_2. If all the weak edges lie on P_1, then P_2 is the strongest $u - v$ path and $\lambda_G(u, v) = s(P_2) = CONN_G(u, v)$. If weak edges lies on both P_1 and P_2, then $\lambda_G(u, v) = 2CONN_G(u, v)$. Thus

$$\lambda_G(u, v) \leq 2CONN_G(u, v), \text{ for each } u, v \text{ in } G$$
$$i.e., \quad \sum_{u,v \in G} \lambda_G(u, v) \leq 2 \sum_{u,v \in G} CONN_G(u, v),$$

and hence $\overline{\lambda}(G) \leq 2$. ■

Theorem 15.2.8. *If a fuzzy graph only contains edges that are β-strong, then $\overline{\lambda}(G) \geq 2$.*

Proof. Consider the fuzzy graph G whose edges are β-strong. So, if uv is an edge in G, then $\mu(uv) = CONN_{G-uv}(u, v)$. Then, the strongest $u - v$ paths include the edge uv and a path of length at least two. Thus there exists at least two strongest $u - v$ paths. That is, $m' \geq 2$ and

$$m' CONN_G(u, v) \geq 2CONN_G(u, v)$$

$$\text{and} \quad \sum_{u,v\in G} m'CONN_G(u,v) \geq \sum_{u,v\in G} 2CONN_G(u,v).$$

Hence, $\overline{\lambda}(G) \geq 2$. ■

For a fuzzy cycle G with G^* a cycle and every edge having the same weight, $\overline{\lambda}(G) = 2$. Also, if $v_1, v_2, \ldots, v_n$ are vertices with corresponding degree sequences $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$ then, $\overline{\lambda}(G) \leq \delta_n$. If the degree of each vertex in a fuzzy graph is at most 3, then the vertex-disjoint and edge-disjoint strongest paths are equal and $\overline{\kappa}(G) = \overline{\lambda}(G)$.

Let $(p_1, p_2, \ldots, p_n)$ be the $n-s$ sequence of a complete fuzzy graph G with l vertices $\{v_1, v_2, \ldots, v_n\}$ and $\sigma(v_i) = p_i$ for $i = 1, 2, \ldots, l$. Theorem 15.2.9 analyzes the AFEC of CFGs with various vertex-strength sequences.

Theorem 15.2.9. *For a complete fuzzy graph $G = (\sigma, \mu)$ with n-s sequence of the form $(p_1^{r_1}, p_2^{r_2}, \ldots, p_{k-1}^{r_{k-1}}, p_k^{r_k})$, we have,*

$$\sum_{u,v\in\sigma^*} \lambda_G(u,v) = \sum_{t=1}^{l-1} (l-t)^2\sigma(v_t) + \sum_{t=1}^{k} p_t R_t$$

where $r_k > 1$, $\sum_{t=1}^{k} r_t = l$ and

$$R_t = \sum_{i=1}^{r_t-1} i(l-(r_{t-1}+\ldots+r_1+1+i))$$

is known as the R_t-factor of the CFG G.

Proof. Let $\sigma^* = \{v_1, v_2, \ldots, v_n\}$ for a CFG G.

Case 1. Suppose $k = l$, $r_i = 1$, $1 \leq i \leq k$, and $p_1 < p_2 < \cdots < p_n$. Consider the edge-disjoint strongest paths between the vertices v_1 and v_2. Since all the edges incident on v_1 has strength p_1, the edge-disjoint strongest $v_1 - v_2$ paths are v_1v_2 and $v_1v_jv_2$ with $j = 3, 4, \ldots, l$. Clearly, all strongest paths between v_1 and v_2 are involved in this and hence there are $l-1$ paths of strength p_1. Thus $\lambda_G(v_1, v_2) = (l-1)\sigma(v_1)$. Hence, for $t = 2, 3, 4, \ldots, l$, we have, $\lambda_G(v_1, v_t) = (l-1)\sigma(v_1)$. Similarly, considering v_2 and v_3, v_2v_3 and $v_2v_jv_3$, $j = 4, 5, \ldots, l$ will be the $l-2$ edge-disjoint v_2v_3 paths with strength p_2. Thus $\lambda_G(v_2, v_t) = (l-2)p_2$, $t = 3, 4, 5, \ldots, l$. Hence, for any $i = 1, 2, \ldots, l$ and $t > i$, $\lambda_G(v_i, v_t) = (l-i)p_i$ as shown in Fig. 15.6. Therefore

$$\sum_{u,v} \lambda_G(u,v) = \sum_{t=1}^{l-1} (l-t)^2\sigma(v_t).$$

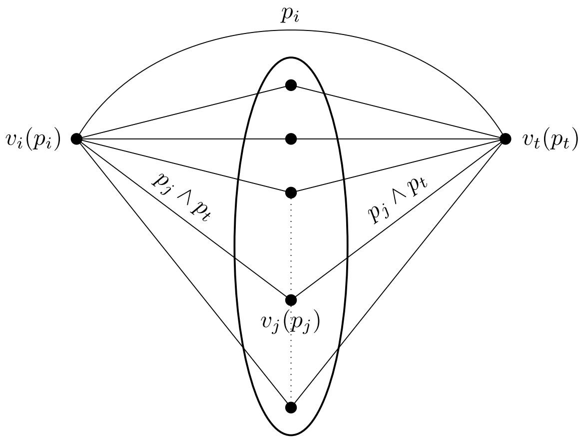

FIGURE 15.6

Edge-disjoint paths in Case 1.

Case 2. Suppose $k = 1$, $r_1 = l$, and $p_1 = p_2 = \cdots = p_n$. Then, for each $i, j = 1, 2, \ldots, l; i \neq j$, there are $(l-1)$ $v_i - v_j$ strongest paths. Hence,

$$\sum_{u,v} \lambda_G(u, v) = (l-1)\sigma(v_1) \sum_{t=1}^{l-1} (l-t).$$

Case 3. Suppose $1 < k < l$, $1 < r_i < l$, $1 \leq i \leq k$, and $p_1 \leq p_2 \leq \cdots \leq p_n$. Let j, $1 < j < k$ be the first integer such that $r_j > 1$. Then for $1 \leq i \leq j$, as in case 1, we have

$$\sum_{t=2}^{l} \lambda_G(v_1, v_t) = (l-1)^2 p_1$$

$$\sum_{t=3}^{l} \lambda_G(v_2, v_t) = (l-2)^2 p_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{t=j+1}^{l} \lambda_G(v_j, v_t) = (l-j)^2 p_j.$$

Now consider the vertex v_{j+1}. For a fixed $t \in \{j+2, \ldots, l\}$, the strongest edge-disjoint $v_{j+1} - v_t$ paths are $v_{j+1}v_t$, $v_{j+1}v_n v_t$, $j+2 \leq l \leq l$, $l \neq t$, and $v_{j+1}v_j v_t$. Then,

$$\sum_{t=j+2}^{l} \lambda_G(v_{j+1}, v_t) = \sum_{t=j+2}^{l} (l-(j+1))\sigma(v_{j+1}) + \sum_{t=j+2}^{l} 1 \times \sigma(v_{j+1})$$
$$= (l-(j+1))^2\sigma(v_{j+1}) + 1 \times (l-(j+1))\sigma(v_{j+1})$$
$$= (l-(j+1))^2\sigma(v_{j+1})$$
$$+ 1 \times (l-(r_{j-1}+\cdots+r_1+1+1))\sigma(v_{j+1})$$

where j in the second term can be rewritten as $j = r_{j-1} + \cdots + r_1 + 1$.

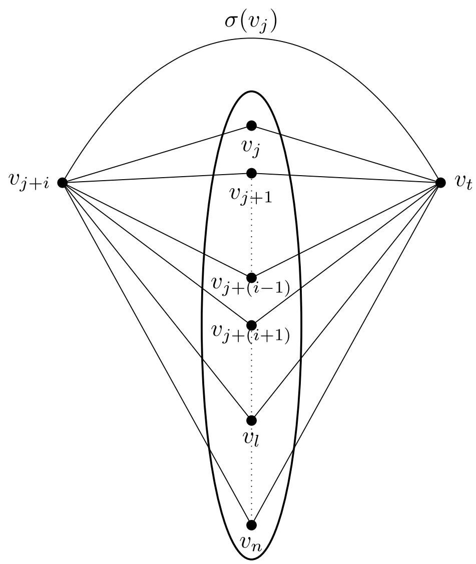

FIGURE 15.7

Edge-disjoint paths in Case 3.

Similarly, consider the vertex v_{j+2}. Proceeding as above, as in Fig. 15.7, the strongest edge-disjoint $v_{j+2} - v_t$ paths for a fixed $t \in \{j+2, \ldots, n\}$ are $v_{j+2}v_t$, $v_{j+2}v_lv_t$, $j+3 \leq l \leq n$, $n \neq t$ together with $v_{j+2}v_jv_t$ and $v_{j+2}v_{j+1}v_t$. Then, $\lambda_G(v_{j+2}, v_t) = \sigma(v_{j+2})(n-(j+2)) + 2\sigma(v_{j+2})$. Thus

$$\sum_{t=j+3}^{n} \lambda_G(v_{j+2}, v_t) = \sum_{t=j+3}^{n} (n-(j+2))\sigma(v_{j+2}) + \sum_{t=j+3}^{n} 2 \times \sigma(v_{j+2})$$
$$= (n-(j+2))^2\sigma(v_{j+2}) + 2 \times (n-(j+3))\sigma(v_{j+2})$$
$$= (n-(j+2))^2\sigma(v_{j+2})$$
$$+ 2 \times (n-(r_{j-1}+\cdots+r_1+1+2))\sigma(v_{j+1}).$$

In general, for $1 \le i \le r_j - 1$,

$$\sum_{t=j+i+1}^{n} \lambda_G(v_{j+i}, v_t) = (n-(j+i))^2\sigma(v_{j+i}) \\ + i \times (n - (r_{j-1} + \cdots + r_1 + 1 + i))\sigma(v_{j+i}).$$

Finally, if $(p_1^{r_1}, p_2^{r_2}, \ldots, p_{k-1}^{r_{k-1}}, p_k^{r_k})$ is the form of the n-s sequence, we can write $\sum \lambda_G(u, v)$ as,

$$\sum_{u,v\in\sigma^*} \lambda_G(u, v) = \sum_{t=1}^{n-1}(n-t)^2\sigma(v_t) + p_1 \sum_{i=1}^{r_1-1} t(n-(1+t)) \\ + p_2 \sum_{i=1}^{r_2-1} t(n-(r_1+1+t)) \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + p_{k-1} \sum_{i=1}^{r_{k-1}-1} t(n-(r_{k-2}+\cdots+r_1+1+t)) \\ + p_k \sum_{i=1}^{r_k-1} t(n-(r_{k-1}+\cdots+r_1+1+t)).$$

Define,

$$R_t = \sum_{i=1}^{r_t-1} i(n-(r_{t-1}+\cdots+r_1+1+i)),\ 1 \le t \le k,$$

as the R_t-factor of the complete fuzzy graph G. Hence,

$$\sum_{u,v\in\sigma^*} \lambda_G(u, v) = \sum_{t=1}^{n-1}(n-t)^2\sigma(v_t) + \sum_{t=1}^{k} p_t R_t$$

where $r_k > 1$, $\sum_{t=1}^{k} r_t = n$. ■

Corollary 15.2.10. *If all the vertices of a complete fuzzy graph G have the same strength, then*

$$\overline{\lambda}(G) = n - 1.$$

Proof. Let $G = (\sigma, \mu)$ be a complete fuzzy graph with vertices $\{v_1, v_2, \ldots, v_n\}$ having strength $\sigma(v_i) = p$ for all $i = 1, 2, \ldots, n$. Then, the n-s sequence of G is (p^n) i.e.

$r_1 = n$ and $p_1 = p$. Hence, from the above theorem,

$$\sum_{u,v\in\sigma^*} \lambda_G(u,v) = \sum_{t=1}^{n-1}(n-t)^2\sigma(v_t) + p_1R_1$$
$$= p\sum_{t=1}^{n-1}[(n-t)^2 + t(n-(1+t)]$$
$$= p\left[\frac{n(n-1)^2}{2}\right].$$

Also, $\sum_{u,v\in\sigma^*} CONN_G(u,v) = \sum_{t=1}^{n-1}(n-t)\sigma(v_t) = p\left[\frac{n(n-1)}{2}\right]$. Hence, $\overline{\lambda}(G) = n - 1$. ■

Let $(p_1^{r_1}, p_2^{r_2}, \ldots, p_{k-1}^{r_{k-1}}, p_k^{r_k})$ be the n-s sequence of a fuzzy graph G with $k = n$, $r_1 = r_2 = \cdots = r_k = 1$, and $p_1 < p_2 < \cdots < p_k$. For the abovementioned case 1 scenario, the AFEC is given by

$$\overline{\lambda}(G) = \frac{\sum_{u,v\in\sigma^*} \lambda_G(u,v)}{\sum_{u,v\in\sigma^*} CONN_G(u,v)} = \frac{\sum_{t=1}^{n-1}(n-t)^2 p_t}{\sum_{t=1}^{n-1}(n-t)p_t}.$$

Example 15.2.11. (Illustration of Theorem 15.2.9) Fig. 15.8 represents a CFG with the n-s sequence $(0.2, 0.3^2, 0.4, 0.6^3, 0.7)$. Table 15.2.1 shows the $u_i - u_j$ edge-disjoint strongest paths and $\lambda_G(u_i,u_j)$ for $1 \le i < j \le 8$.

We have, $\sum_{u,v\in\sigma^*} \lambda_G(u,v) = 46.8$ and $\sum_{u,v\in\sigma^*} CONN_G(u,v) = 9.9$. Hence, we get the AFEC $\overline{\lambda}(G) = 4.72$.

Theorem 15.2.12 gives the relation between fuzzy edge connectivity and the average fuzzy edge connectivity of fuzzy graphs.

Theorem 15.2.12. *For a fuzzy graph G, $\kappa'(G) \le \overline{\lambda}(G)$.*

Proof. Let S be a fuzzy edge cut with strong weight, $s'(S) = \sum_{s_i\in S}\mu(s_i)$. Then there exists $u, v \in \sigma^*$ such that either $CONN_{G-S}(u,v) < CONN_G(u,v)$ or $G - S$ is disconnected. Then, by Menger's Theorem, Theorem 4 in [65],

$$\sum_{s_i\in S}\mu(s_i)CONN_G(u,v) \le |S|CONN_G(u,v)$$
$$\le m'_{u,v}CONN_G(u,v)$$

Table 15.2.1 Table in Example 15.2.11.

u_i	u_j	Strongest $u_i - u_j$ paths	$\kappa_G(u_i, u_j)$
u_1	u_2	u_1u_2, $u_1u_3u_2$, $u_1u_4u_2$, $u_1u_5u_2$, $u_1u_6u_2$, $u_1u_7u_2$, $u_1u_8u_2$	$7 \times 7 \times 0.2$
	u_3	u_1u_3, $u_1u_2u_3$, $u_1u_4u_3$, $u_1u_5u_3$, $u_1u_6u_3$, $u_1u_7u_3$, $u_1u_8u_3$	
	u_4	u_1u_4, $u_1u_2u_4$, $u_1u_3u_4$, $u_1u_5u_4$, $u_1u_6u_4$, $u_1u_7u_4$, $u_1u_8u_4$	
	u_5	u_1u_5, $u_1u_2u_5$, $u_1u_3u_5$, $u_1u_4u_5$, $u_1u_6u_5$, $u_1u_7u_5$, $u_1u_8u_5$	
	u_6	u_1u_6, $u_1u_2u_6$, $u_1u_3u_6$, $u_1u_4u_6$, $u_1u_5u_6$, $u_1u_7v_6$, $u_1u_8u_6$	
	u_7	u_1u_7, $u_1u_2u_7$, $u_1u_3u_7$, $u_1u_4u_7$, $u_1u_5u_7$, $u_1u_6u_7$, $u_1u_8u_7$	
	u_8	u_1u_8, $u_1u_2u_8$, $u_1u_3u_8$, $u_1u_4u_8$, $u_1u_5u_8$, $u_1u_6u_8$, $u_1u_7u_8$	
u_2	u_3	u_2u_3, $u_2u_4u_3$, $u_2u_5u_3$, $u_2u_6u_3$, $u_2u_7u_3$, $u_2u_8u_3$	$6 \times 6 \times 0.3$
	u_4	u_2u_4, $u_2u_3u_4$, $u_2u_5u_4$, $u_2u_6u_4$, $u_2u_7u_4$, $u_2u_8u_4$	
	u_5	u_2u_5, $u_2u_3u_5$, $u_2u_4u_5$, $u_2u_6u_5$, $u_2u_7u_5$, $u_2u_8u_5$	
	u_6	u_2u_6, $u_2u_3u_6$, $u_2u_4u_6$, $u_2u_5u_6$, $u_2u_7u_6$, $u_2u_8u_6$	
	u_7	u_2u_7, $u_2u_3u_7$, $u_2u_4u_7$, $u_2u_5u_7$, $u_2u_6u_7$, $u_2u_8u_7$	
	u_8	u_2u_8, $u_2u_3u_8$, $u_2u_4u_8$, $u_2u_5u_8$, $u_2u_6u_8$, $u_2u_7u_8$	
u_3	u_4	u_3u_4, $u_3u_2u_4$, $u_3u_5u_4$, $u_3u_6u_4$, $u_3u_7u_4$, $u_3u_8u_4$	$5 \times 6 \times 0.3$
	u_5	u_3u_5, $u_3u_2u_5$, $u_3u_4u_5$, $u_3u_6u_5$, $u_3u_7u_5$, $u_3u_8u_5$	
	u_6	u_3u_6, $u_3u_2u_6$, $u_3u_4u_6$, $u_3u_5u_6$, $u_3u_7u_6$, $u_3u_8u_6$	
	u_7	u_3u_7, $u_3u_2u_7$, $u_3u_4u_7$, $u_3u_5u_7$, $u_3u_6u_7$, $u_3u_8u_7$	
	u_8	u_3u_8, $u_3u_2u_8$, $u_3u_4u_8$, $u_3u_5u_8$, $u_3u_6u_8$, $u_3u_7u_8$	
u_4	u_5	u_4u_5, $u_4u_6u_5$, $u_4u_7u_5$, $u_4u_8u_5$	$4 \times 4 \times 0.4$
	u_6	u_4u_6, $u_4u_5u_6$, $u_4u_7u_6$, $u_4u_8u_6$	
	u_7	u_4u_7, $u_4u_5u_7$, $u_4u_6u_7$, $u_4u_8u_7$	
	u_8	u_4u_8, $u_3u_5u_8$, $u_3u_6u_8$, $u_3u_7u_8$	
u_5	u_6	u_5u_6, $u_5u_7u_6$, $u_5u_8u_6$	$3 \times 3 \times 0.6$
	u_7	u_5u_7, $u_5u_6u_7$, $u_5u_8u_7$	
	u_8	u_5u_8, $u_5u_6u_8$, $u_5u_7u_8$	
u_6	u_7	u_6u_7, $u_6u_5u_7$, $u_6u_8u_7$	$2 \times 3 \times 0.6$
	u_8	u_6u_8, $u_6u_5u_8$, $u_6u_7u_8$	
u_7	u_8	u_7u_8, $u_7u_5u_8$, $u_7u_6u_8$	$1 \times 3 \times 0.6$

where $m'_{u,v}$ is the number of edge-disjoint strongest paths between u and v. Subsequently, for all $u, v \in G$,

$$\sum_{u,v \in G} \sum_{s_i \in S} \mu(s_i) CONN_G(u, v) \leq \sum_{u,v \in G} m'_{u,v} CONN_G(u, v).$$

Since $\kappa'(G)$ is the minimum strong weight of fuzzy edge cuts we have,

$$\kappa'(G) \sum_{u,v \in G} CONN_G(u, v) \leq \sum_{u,v \in G} m'_{u,v} CONN_G(u, v).$$

Hence, $\kappa'(G) \leq \overline{\lambda}(G)$. ■

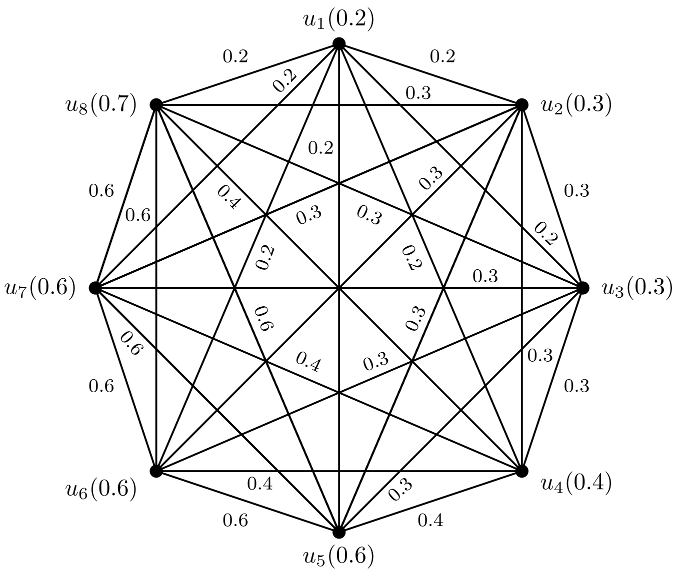

FIGURE 15.8

Complete fuzzy graph in Example 15.2.11.

Uncertainty exists regarding the disproportion between AFVC $\overline{\kappa}(G)$ and fuzzy edge connectivity $\kappa'(G)$. From [2], we get $\kappa(G) \leq \overline{\kappa}(G)$. For the fuzzy graph given in Fig. 15.3, $X = \{v_2v_4\}$ is a fuzzy edge cut with minimum strong weight $\kappa'(G) = 0.1$ and $\overline{\kappa}(G) = 1.14$. But, for the fuzzy graph in Fig. 15.2, $Y = \{v_1v_4, v_3v_4\}$ is a fuzzy edge cut with minimum strong weight $\kappa'(G) = 0.2$ and $\overline{\kappa}(G) = 1.8$. Consequently, we may only state that

$$\kappa(G) \leq \kappa'(G) \leq \overline{\lambda}(G),$$
$$\kappa(G) \leq \overline{\kappa}(G) \leq \overline{\lambda}(G).$$

Let us examine the sufficient requirement for the inequality of $\overline{\kappa}(G)$ and $\overline{\lambda}(G)$.

Theorem 15.2.13. *For a fuzzy graph $G = (\sigma, \mu)$, if $\kappa_G(u, v) < \lambda_G(u, v)$, for some $u, v \in \sigma^*$ then there exists an edge-disjoint fuzzy subgraph containing a fuzzy cycle of strength $CONN_G(u, v)$.*

Proof. Suppose for a fuzzy graph G, there exist two vertices u, v in G such that $\kappa_G(u, v) < \lambda_G(u, v)$. That is, there exist at least two edge-disjoint strongest u, v paths which are not vertex disjoint. Consider two paths P_1 and P_2, which are edge disjoint but not vertex disjoint. Let us assume that P_1 and P_2 encounter each other at v_1 and v_2, respectively, as illustrated in Fig. 15.9. The paths P_1 and P_2 induces a subgraph, which is a union of edge-disjoint cycles. Let the cycles be denoted by C_1, C_2, and C_3. Since edges of C_1, C_2, and C_3 are edges of paths P_1 and P_2, the strongest

$u - v$ paths, the strength of each cycle is at least $CONN_G(u, v)$. Additionally, at least one cycle has strength $CONN_G(u, v)$. Otherwise, edges of each cycle have weights greater than $CONN_G(u, v)$, which is contradictory.

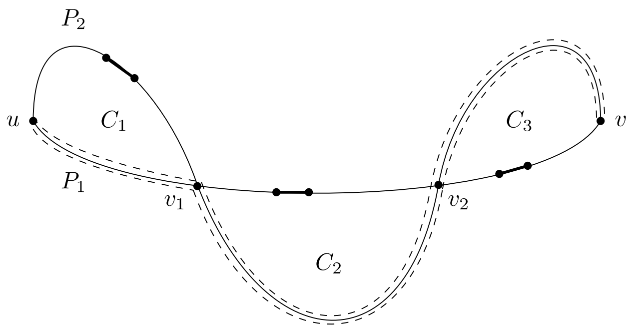

FIGURE 15.9

Fuzzy graph in Theorem 15.2.13.

We claim that C_1, C_2, or C_3 is a fuzzy cycle. If this is not the case, assume that each cycle in the induced fuzzy subgraph has precisely one weakest edge. Let the weakest edges be denoted by thick lines. The $u - v$ path that results from merging the paths without the weakest edge, as illustrated in Fig. 15.9 by dotted lines, will then have a strength greater than $CONN_G(u, v)$, which is a contradiction. As a result, at least one cycle has at least two weak edges, forming a fuzzy cycle.

Now, P_1 and P_2 being strongest, there is at least one edge with a weight of $CONN_G(u, v)$ on P_1 and P_2. A fuzzy cycle with strength $CONN_G(u, v)$ will exist if at least one of the weakest edges of P_1 and P_2 is found in the same cycle. If not, any $u - v$ path that does not contain the weakest edges of P_1 and P_2 will then be a path with strength greater than $CONN_G(u, v)$. Hence, at least one of C_1, C_2, or C_3 is a fuzzy cycle of strength $CONN_G(u, v)$. As a result, we conclude that there exists a fuzzy subgraph containing a fuzzy cycle with strength $CONN_G(u, v)$. ■

The converse of the preceding theorem may not hold true in all scenarios. Consider a fuzzy cycle $G = (\sigma, \mu)$ with strength equal to $\mu(uv) = \text{CONN}_G(u, v)$ for some edge $uv \in \mu^*$. In this case, G constitutes an edge-disjoint subgraph of itself, containing a fuzzy cycle with strength $\text{CONN}_G(u, v)$. However, for a fuzzy cycle, $\kappa_G(u, v) = \lambda_G(u, v)$ for all $u, v \in \sigma^*$.

15.3 Application

Children are deeply impacted by the events that take place around them. Even though they may not understand what they see and hear, they absorb and are affected by the people they rely on for love and security. Constant unrelenting negative experiences—known as “toxic stress”—take a toll on a child’s growth and development. In response to stress, the brain produces the hormone cortisol. During prolonged exposure to stress, cortisol levels remain too high for too long, which inhibits brain development. Over time, this can change the architecture of a child’s rapidly developing brain development. Altered brain architecture can result in long-term problems in learning, behavior, and physical and mental health, [112].

Among the more common disorders that can be diagnosed in childhood are attention-deficit/hyperactivity disorder (ADHD), anxiety, and behavior disorders as well as depression. Each of these conditions can significantly impact a child’s life, and they often interact with each other, leading to complex and varied experiences. For instance, a child with anxiety might also exhibit symptoms of depression, or a behavior disorder might be accompanied by anxiety. We first concentrate on anxiety (A), behavior (B), depression (D), Mental Disorder (Md), and Development Disorder (Dd). We construct a fuzzy graph involving these disorders as the vertices as well weighted graphs associated with their connection. The values are taken from [113].

We illustrate how m', λ_G and the deletion of an edge can be used to understand the structure of a fuzzy graph. Considering the following fuzzy graph $G = (V, E)$ given in Fig. 15.10, we get Table 15.3.1.

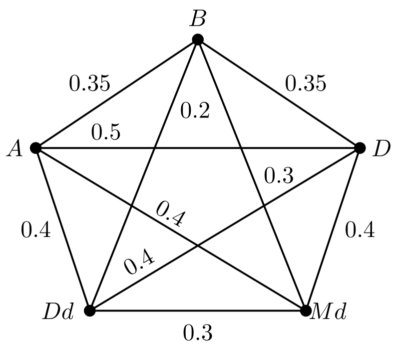

FIGURE 15.10

Fuzzy graph G.

Table 15.3.1 Table representing connectivity, m and m' of fuzzy graph M.

Vertices	A, B	A, D	B, D	A, Md	B, Md	D, Md	A, Dd	B, Dd	D, Dd	Md, Dd
$CONN$	0.35	0.5	0.35	0.4	0.35	0.4	0.4	0.35	0.4	0.4
m'	2	1	2	2	2	2	2	2	2	2

It follows that $\sum_{u,v} \lambda_G(u, v) = 7.3$ and $\sum_{u,v} CONN_G(u, v) = 3.9$. Hence

$$\frac{\sum_{u,v} \lambda_G(u, v)}{\sum_{u,v} CONN_G(u, v)} = \frac{7.3}{3.9} = 1.87.$$

We next delete the edge AD. We obtain Table 15.3.2 with edge AD deleted. Thus

Table 15.3.2 Table representing connectivity, m and m' of fuzzy graph $G - AD$.

Vertices	A, B	A, D	B, D	A, Md	B, Md	D, Md	A, Dd	B, Dd	D, Dd	Md, Dd
CONN	0.35	0.4	0.35	0.4	0.35	0.4	0.4	0.35	0.4	0.4
m'	2	2	2	2	2	1	2	2	2	2

$\sum_{u,v} \lambda_{G-AD}(u, v) = 7.6$ and $\sum_{u,v} CONN_{G-AD}(u, v) = 3.8$. Hence,

$$\frac{\sum_{u,v} \lambda_{G-AD}(u, v)}{\sum_{u,v} CONN_{G-AD}(u, v)} = \frac{7.6}{3.8} = 2.$$

The increase from 1.87 to 2 shows that the absence of Anxiety(A)- Depression(D) relationship will leads to stronger influences among the other conditions.

Now, consider the edge ADd. We obtain Table 15.3.3 upon removal of the edge ADd.

Table 15.3.3 Table representing connectivity, m and m' of fuzzy graph $G - ADd$.

Vertices	A, B	A, D	B, D	A, Md	B, Md	D, Md	A, Dd	B, Dd	D, Dd	Md, Dd
CONN	0.35	0.5	0.35	0.4	0.35	0.4	0.4	0.35	0.4	0.4
m'	2	1	2	2	2	2	1	1	1	1

Thus $\sum_{u,v} \lambda_{G-ADd}(u, v) = 5.75$ and $\sum_{u,v} CONN_{G-ADd}(u, v) = 3.9$. Hence,

$$\frac{\sum_{u,v} \lambda_{G-ADd}(u, v)}{\sum_{u,v} CONN_{G-ADd}(u, v)} = \frac{5.75}{3.9} = 1.47.$$

Here, there is a decrease in $\overline{\lambda}$, shows that much concentration has to be done on the Anxiety-Development disorder connection. It helps in understanding the relative influence or impact of certain connections on the network's structure and resilience to changes or disruptions.

A higher average value would indicate a more interconnected or strongly correlated network of conditions, while a lower value suggests weaker overall interactions. Understanding these shifts informs adjustments in clinical strategies, guides further research to explore emerging relationships and highlights the adaptability of mental health networks in response to changes in connections between conditions. Recognizing the decrease in AFEC after removing a specific relationship (edge) highlights its importance in the network. Exposure to many adverse childhood experiences can be found in [106]. We present this application as a first step in using fuzzy mathematics in the study of toxic stress among children.

Exposure to many adverse childhood experiences can be found in [106]. We present this application as a first step in using fuzzy mathematics in the study of toxic stress among children.

15.4 Exercises

1. Calculate the AFVC of the fuzzy graph $G = (\sigma, \mu)$ where $\sigma^* = \{v_1, v_2, ..., v_8\}$ $\sigma(v) = 1$ for every $v \in \sigma^*$, $\mu(v_1 v_2) = 0.1$, $\mu(v_2 v_3) = 0.2$, $\mu(v_3, v_4) = 0.3$, $\mu(v_4, v_5) = 0.4$, $\mu(v_5, v_6) = 0.5$, $\mu(v_6, v_7) = 0.6$, $\mu(v_7, v_8) = 0.7$, $\mu(v_8 v_1) = 0.8$.

2. Find the AFVC of $G = (\sigma, \mu)$ with $\sigma^* = \{v_1, v_2, v_3, v_4, v_5\}$, $\sigma(v) = 1$ for every $v \in \sigma^*$ $\mu(v_3 v_4) = \mu(v_4 v_5) = 0.3$ and $\mu(v_1 v_2) = \mu(v_1 v_5) = \mu(v_2 v_5) = \mu(v_3 v_5) = 0.5$.

Bibliography

[1] ACLU, Human trafficking: modern slavery enslavement of immigrant women in the United States, https://www.aclu.org/other/human-trafficking-modern-enslavement-immigrant-women-united-states.

[2] S. Ali, S. Mathew, J.N. Mordeson, H. Rashmanlou, Vertex connectivity of fuzzy graphs with applications to human trafficking, New Math. Nat. Comput. 14 (2018) 457–485.

[3] A. Koop, Visualized: which countries are dominating space?, https://visualcapitalist.com/visualised-which-countries-are-dominating-space/, July 2022 (Graphic/Design: Sabrina Fortin).

[4] A.A. Aboulatta, 2030 Agenda - a unique opportunity to address conditions conducive to the spread of terrorism, UN chronicle, implementing the 2030-Agenda, Challenge Confl. LII (4) (April 2015) 1–5.

[5] G.A. Ackerman, Kevin S. Moran, Bioterrorism and Threat Assessment, The Weapons of Mass Destruction Commission, 2004, pp. 3–6.

[6] S. Akash, InQubeta, the 100x AI Crypto Token, Top 10 Internet Scamming Countries in the World in 2023, March 2023.

[7] P. Aswathy, S. Mathew, J.N. Mordeson, Generalized cycle connectivity in fuzzy graphs, Fuzzy Sets Syst. 482 (11) (2024) 108909.

[8] K.T. Atannasov, Intuitionistic Fuzzy Sets, Theory and Applications, Studies in Fuzziness and Soft Computing, Springer, vol. 35, Physica Verlag, 1999.

[9] W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication, Fuzzy Sets Syst. 4 (1980) 13–30.

[10] L. Beineke, O. Oellermann, R. Pippert, The average connectivity of a graph, Discrete Math. 252 (2002) 31–45.

[11] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognit. Lett. 6 (1987) 297–302.

[12] K.R. Bhutani, On automorphisms of fuzzy graphs, Pattern Recognit. Lett. (1989) 159–162.

[13] K.R. Bhutani, A. Rosenfeld, Fuzzy end nodes in fuzzy graphs, Inf. Sci. 152 (2003) 323–326.

[14] K.R. Bhutani, A. Rosenfeld, Strong arcs in fuzzy graphs, Inf. Sci. 152 (2003) 319–322.

[15] M. Binu, S. Mathew, J.N. Mordeson, Connectivity index of a fuzzy graph and its application to human trafficking, Fuzzy Sets Syst. 360 (2018) 117–136.

[16] M. Binu, S. Mathew, J.N. Mordeson, Cyclic connectivity index of fuzzy graphs, IEEE Trans. Fuzzy Syst. 29 (6) (2021) 1340–1349.

[17] M. Binu, S. Mathew, J.N. Mordeson, Wiener index of a fuzzy graph and application to illegal immigration networks, Fuzzy Sets Syst. 384 (2019) 132–147.

[18] R.C. Brigham, P.Z. Chinn, R.D. Dutton, Vertex dominationcritical graphs, Networks 18 (3) (1988) 173–179.

[19] A. Buttars, Global Cybercrime Risk rankings, Quest, A Journal of Undergraduate Student Research.

[20] J. Burg, 6 common uses of AI in telecommunications 2023, TechSee, https://techse.me/blog/artificial-intelligence-in-telecommunications-industry/.

[21] Children's Mental Health Data, Statistics in Children's Mental Health, pp. 1–4.

[22] Clear Air Fund, Sustainable development goals and air pollution, Blog https://www.cleanairfund.org/news-item/sustainable-development, 12 July 2023.
[23] Committee on the Peaceful Uses of Outer Space, Guidelines for the long-term sustainability of outer space activities, Vienna, 20–29 June 2018.
[24] Country-Network Readiness, Index, Countries, benchmarking the future of the network economy, University of Oxford, Said Business School, 2023, networkreadinessindex.org/countries.
[25] Countries Ranking by Health Care Index in 2023 - Take-profit.org, https://take-profit.org/en/statistics/health-care-index/.
[26] M. Delgado, J. Verdegay, M. Vila, On fuzzy tree definition, Eur. J. Oper. Res. 22 (2) (1985) 243–249.
[27] A.P. Dempster, Upper and lower probabilities induced by multivalued mappings, Ann. Math. Stat. 38 (2) (1967) 325–339.
[28] A.P. Dempster, Upper and lower probability inferences based on a sample from a finite univariate population, Biometrika 54 (3,4) (1967) 515–528.
[29] J. Dombi, On implication operators, Advances in Intelligent Systems and Computing, 2018.
[30] Facts and figures: Ending violence against women, UN Women-Headquarters, pp. 1–12, http://www.unwomen.org/en/what-we-do/ending-violence-against-women/facts-and-figures.
[31] G. Falco, Cybersecurity principles for space systems, J. Aerosp. Inform. Syst. 16 (2) (2019), https://doi.org/10.2514/1.I010693.
[32] A. Fleck, Who's responsible for space junk, Statista (Sep. 2022).
[33] A.N. Gani, M.B. Ahamed, Order and size in fuzzy graph, Bull. Pure Appl. Sci. 22 (1) (2003) 145–148.
[34] A.N. Gani, K. Radha, The degree of a vertex in some fuzzy graphs, Int. J. Algorithms Comput. Math. 2 (2009).
[35] A.N. Gani, J. Malarvizhi, Isomorphism on fuzzy graphs, Int. J. Comput. Sci. Math. 2 (2008) 200–206.
[36] Gender-based violence creates an unequal world for children, Save the Children, https://www.savechildren.org/us/charity-stories/gender-based-violence, pp. 1–5.
[37] Gender Inequality Index, Wikipedia https://en.wikipedia.org/wiki/Gender_Inequality_Index, pp. 1–11.
[38] Global Cybersecurity Index 2020, measuring commitment to cybersecurity, International Telecommunication Union.
[39] 2021 Global Health Security Index, John Hopkins Center for Health Security, https://www.ghsindex.org.
[40] Global Health Security Index, 2016, http://www.globalslaveryindex.org/findings.
[41] Global Slavery Index 2018, the Walk Free Foundation.
[42] Global Terrorism, Index 2023, Measuring the impact of terrorism, IES Institute for Economics and Peace.
[43] I. Gutman, M. Togan, A. Yurttas, A.S. Cevik, I.N. Cangul, Inverse problem for sigma index, MATCH Commun. Math. Comput. Chem. 79 (2) (2018) 491–508.
[44] S.R. Islam, M. Pal, First Zagreb index on a fuzzy graph and its application, J. Intell. Fuzzy Syst. 40 (6) (2021) 10575–10587.
[45] S.R. Islam, M. Pal, Second Zagreb index for fuzzy graphs and its application in mathematical chemistry, Iran. J. Fuzzy Syst. 20 (1) (2023) 119–136.
[46] Internet security threat report archive, Symantec Corporation, http://www.symantec.com/threatreport/archive.jsp/.

[47] INTERPOL, Global policing goals, https://www.interpol.int/en/Who-we-are/Strategy/Global-Policing-Goals.
[48] G. Larson, At-Risk Countries for bioterrorism: an assessment Method for determining a country's general vulnerability to a bioterrorism attack, Quest: A Journal of Undergraduate Student Research, 62–109.
[49] O.C. Havare, H. Menken, On corona product of two fuzzy graphs, Int. J. Fuzzy Math. Archive 10 (2) (2016) 95–100.
[50] N. Jicy, S. Mathew, Connectivity analysis of cyclically balanced fuzzy graphs, Fuzzy Inf. Eng. 7 (2015) 245–255.
[51] J. Fong, Global Cybercrime Report: Which countries are most at risk 2023? SEON.
[52] A. Joshi, Space junk used for cyber warfare?, Space Syst. CyberSecurity (October 2022), https://spacesecurity.wse.jhu.edu/2022/10/26/unhealthy-space-space-junks/.
[53] A. Josy, S. Mathew, J.N. Mordeson, Neighborhood connectivity index of a fuzzy graph and its application to human trafficking, Iran. J. Fuzzy Syst. 19 (3) (2022) 139–154.
[54] A. Josy, S. Mathew, J.N. Mordeson, Fuzzy graph parameters: Applied to the study of illegal migration to the United States, Fuzzy Information and Engineering, 2024, in press.
[55] A. Josy, S. Mathew, J.N. Mordeson, First and second K Banhatti indices of a fuzzy graph applied to decision making, Inf. Sci. 678 (2024) 120961.
[56] E.P. Klement, R. Mesiar, L-fuzzy sets and isomorphic lattices: are the new results really new?, Mathematics 6 (2018) 146.
[57] A. Kauffmann, Introduction to the Theory of Fuzzy Sets, vol. 1, Academic Press, Inc., Orlando, Florida, 1973.
[58] J. Kim, O. Suil, Average connectivity and average edge-connectivity in graphs, Discrete Math. 313 (2013) 2232–2238.
[59] V.R. Kulli, On K Banhatti indices of graphs, J. Comput. Math. Sci. 7 (4) (2016) 213–218.
[60] G.D. Kyriakopoulos, P. Pazartzis, A. Koskina, C. Bourcha, Artificial intelligence and space situational awareness: data processing and sharing in debris-crowded areas, in: Proc. 8th European Conference on Space Debris (virtual) Darnstach Germany, 20–23 April, 2021, published by the FSA Space Debris Office.
[61] List of countries by natural disaster risk, Wikipedia, the Free Encyclopedia, World Conference on Disaster Risk Reduction, https://en.wikeipedia.org/wiki/List_of_countries_by_natural_disaster_risk.
[62] S. Mathew, M.S. Sunitha, Types of arcs in a fuzzy graph, Inf. Sci. 179 (11) (2009) 1760–1768.
[63] S. Mathew, M.S. Sunitha, Node connectivity and arc connectivity of a fuzzy graph, Inf. Sci. 180 (2010) 519–531.
[64] S. Mathew, M.S. Sunitha, Cycle connectivity in fuzzy graphs, J. Intell. Fuzzy Syst. 24 (2013) 549–554.
[65] S. Mathew, M.S. Sunitha, Menger's theorem for fuzzy graphs, Inf. Sci. 222 (2013) 717–726.
[66] S. Mathew, H.L. Yang, J.K. Mathew, Saturation in fuzzy graphs, New Math. Nat. Comput. 14 (01) (2018) 113–128.
[67] D.W. Matula, Cluster analysis via graph theoretic techniques, in: Proceedings of the Louisiana Conference on Combinatorics, Graph Theory, and Computing, March 1970, pp. 199–212.
[68] J.N. Mordeson, Fuzzy line graphs, Pattern Recognit. Lett. 14 (1993) 381–384.

[69] J.N. Mordeson, S. Mathew, Non-deterministic flow in fuzzy networks and its application in identification of human trafficking chains, New Math. Nat. Comput. 13 (2017) 231–243.
[70] J.N. Mordeson, S. Mathew, D. Malik, Fuzzy Graph Theory with Applications to Human Trafficking, Springer, 2018.
[71] J.N. Mordeson, S. Mathew, Mathematics of Uncertainty for Coping with World Challenges: Climate Change, World Hunger, Modern Slavery, Coronavirus, Human Trafficking, Studies in Systems, Decision and Control, vol. 353, Springer, 2021.
[72] J.N. Mordeson, S. Mathew, Sustainable Development Goals: Analysis by Mathematics of Uncertainty, Studies in Systems, Decision and Control, vol. 299, Springer, 2021.
[73] J.N. Mordeson, S. Mathew, Similarity of country rankings on sustainability performance, Trans. Fuzzy Sets Syst. 2 (1) (2023) 1–21.
[74] J.N. Mordeson, S. Mathew, Terrorism: interpol policing goals and SDGs, New Math. Nat. Comput. (2024), in press.
[75] J.N. Mordeson, S. Mathew, Biodiversity: SDGs and Aichi targets, J. Mahani Math. Res. Cent. (2024), in press.
[76] J.N. Mordeson, S. Mathew, Fuzzy implication operators: health security and political risk, New Math. Nat. Comput. (2024), in press.
[77] J.N. Mordeson, S. Mathew, t-norm fuzzy graphs, New Math. Nat. Comput. 14 (1) (2018) 129–143.
[78] J.N. Mordeson, S. Mathew, P. Aswathy, Fuzzy implication operators applied to country health preparation, Trans. Fuzzy Sets Syst. 3 (1) (2024) 57–66.
[79] S. Mathew, J.N. Mordeson, D. Malik, Fuzzy Graph Theory, Springer, 2018.
[80] J.N. Mordeson, S. Mathew, D.S. Malik, Generalized fuzzy relations, New Math. Nat. Comput. 14 (2) (2018) 187–202.
[81] J.N. Mordeson, S. Mathew, P. Sujithra, Mistreatment of women and children: analysis by mathematics of uncertainty, J. Alg. Hyperstruct. Log. Alg. (2024), in press.
[82] J.N. Mordeson, P.S. Nair, Fuzzy Graphs and Fuzzy Hypergraphs, Physica - Verlag, 2000.
[83] J.N. Mordeson, P.S. Nair, Cycles and cocycles of fuzzy graphs, Inf. Sci. 90 (1996) 39–49.
[84] J.N. Mordeson, H.C. Wething, T.C. Clark, A fuzzy mathematical model of nuclear stability, New Math. Nat. Comput. 6 (2010) 119–140.
[85] M.J. Morgan, S. Mukwembi, H.C. Swart, On the eccentric connectivity index of a graph, Discrete Math. 311 (13) (2011) 1229–1234.
[86] Nord VPN Cyber Risk Index, 2020.
[87] Political Stability - Country rankings, 2023 Global Threat Report, https://www.theglobaleconomy.com/rankings/wb_political_stability.
[88] Political Risk (September 2021), Political Risk Index (September 2021) - PRS Group, https://www.prsgroup.com/political-risk-index-september-2021.
[89] G. Ramsey, Routes to the US: Mapping Human Smuggling Networks, 2011.
[90] Ranking of countries according to their respect for the Rights of the Child, Humanium, https://www.humanium.org/en/.
[91] Routes to the united U.S.: Mapping Human Smuggling Networks, http://www.insightcrime.org/newsanalysis/routes-to-us-mapping.
[92] A. Rosenfeld, Fuzzy graphs, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York, 1975, pp. 77–95.
[93] T.L. Saaty, A scaling method for priorities in hierarchical structures, J. Math. Psychol. 15 (1977) 234–281.

[94] T.L. Saaty, The Analytic Hierarchy Process, McGraw Hill, York, 1980.
[95] T.L. Saaty, Relative measurement and its generalization in decision making: why pairwise comparisons are central in mathematics for the measurement of intangible factors, the analytic hierarchy process, RACSAM Rev. R. Span. Acad. Sci. Ser. A, Math. 103 (2) (2008) 252–319.
[96] K. Sameena, M.S. Sunitha, A characterization of g-selfcenterd fuzzy graphs, J. Fuzzy Math. 6 (4) (2008) 787–791.
[97] S. Desire, N.O. Mediehie, A policy dialogue for nutrition of women and adolescent girls: sustainable development goals matters arising, Oeios (Nov. 2023) 1–6.
[98] Scam Alert, Top 10 scamming countries in the world to be aware.
[99] SDG Index and Dashboard Report 2017, Global Responsibilities, International Spillovers in Achieving the Goals, Bertelsmann Stiftung, Sustainable Development Solutions Network, 2017.
[100] A. Sebastian, S. Mathew, J.N. Mordeson, Generalized fuzzy graph connectivity parameters with application to human trafficking, Mathematics 8 (3) (2020) 424–444.
[101] P. Shah, Visualizing the world's Space Debris by Country Responsible, Aug. 4, 2023 (Article/Editing Freny Fernandes).
[102] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, Princeton, 1976.
[103] V. Sharma, R. Goswami, A.K. Madan, Eccentric connectivity index: a novel highly discriminating topological descriptor for structure-property and structure-activity studies, J. Chem. Inf. Comput. Sci. 37 (1997) 273–282.
[104] F. Smarandach, Neutrosophy: Neutrosophic Probability. Set and Logic: Analytic Synthesis and Synthetic Analysis, American Research Press, Rehoboth, NM, 1998.
[105] Speediest Global Index-Internet-Speed Around the World-Speediest Global Index, https://www.speediest.net/global-index#mobile.
[106] Stress in Early Childhood, Wikipedia, https://en.wikipedia.org/wiki/Stress_in_early_childhood.
[107] P. Sujithra, M. Mathew, J.N. Mordeson, Average fuzzy edge connectivity of fuzzy graphs with applications to toxic stress in youth, in: Fuzzy Sets and Systems, submitted for publication.
[108] M.S. Sunitha, A. Vijayakumar, Complement of a fuzzy graph, Indian J. Pure Appl. Math. 33 (9) (2002) 1451–1464.
[109] Sustainable Development Report 2019 Transformations to Achieve the Sustainable Developmental Goals, SDG Index and Dashboards.
[110] Telecommunications Output, This key economic indicator for the Telecom sector has been recently updated, https://wwwnationmaster.com/nms/ranking/telecommunications-output.
[111] Tortoise, The global AI index, https://www.tortoisemedia.com/intelligence/global-ai/.
[112] Toxic Stress, Children's lifetime, stress and trauma, https://developingchild.harvard.edu/science/key-concepts/toxic-stress/, 2019.
[113] Toxic Stress, Data and statistics on children's mental health, https://www.cdc.gov/childrensmentalhealth/data.html, 2019.
[114] Tracking sustainable peace through inclusion, justice, and security for women, Women Peace and Security Index, Georgetown Institute for Women Peace and Security Index (DGIEWIS), pp. 1–82.
[115] Trafficking in Persons: Global Patterns, the United Nations Office for Drugs and Crime, Trafficking Persons Citation Index.

[116] E. Trillas, I. Garcia-Honrado, A reflection on dialectic synthesis, New Math. Nat. Comput. 15 (1) (2019) 31–46.
[117] E. Trillas, A.R. de Soto, On the search for speculations, New Math. Nat. Comput. 18 (1) (2022) 9–18.
[118] E. Trillas, A.R. de Soto, On a new view of a fuzzy set, Trans. Fuzzy Sets Syst. 2 (1) (2023).
[119] Truecaller, 2021 Global Spam and Scam Report, Truecaller Insights: Top 20 countries affected by spam calls in 2021, Dec 17, 2021.
[120] Understanding Children Rights, Humanium, pp. 1–4, http://www.humanium.org/en/fundamental-rights/.
[121] The Walk Free Foundation, The Global Slavery Index 2016, 2016.
[122] X. Wang, B. De Baets, E. Kerre, A comparative study of similarity measures, Fuzzy Sets Syst. 73 (1995) 259–268.
[123] Wikipedia, Space competitive index, https://en.wikipedia.org/wiki/Space_Competitiveness_Index (SCI.Corporation%2C a U.S. consulting firm).
[124] World Economic Forum, What's the environmental impact of space debris and how can we solve it?, https://www.weforum.org/agenda/2022/07/environmental-impact-space-debris-how-to-solve-it/, July 2022.
[125] R.R. Yager, Pythagorean membership grades in multi-criteria decision making, IEEE Trans. Fuzzy Syst. 22 (2014) 958–965.
[126] R.T. Yeh, S.Y. Bang, Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, 1975, pp. 125–149.
[127] J. Yen, Generalizing the Dempster-Shafer theory to fuzzy sets, in: Z. Wang, G.J. Klir (Eds.), Fuzzy Measure Theory, Plenum Press, New York, 1992, pp. 257–283, Ch. 7.
[128] Youth Homelessness Overview, NCSL 1-2, Copyright 2022 by National Conference of State Legislature.
[129] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.

Index

M

N

O

P

R

S